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Nine Out of Ten Like Consumer Credit Problems

By LEBARON R. FOSTER

Opinion Research Corporation, Princeton, N.J.

IN ALL schools, large or small in size, high or low in rating, recent tests show that much of the knowledge that teachers pour on students does not soak in. It rolls off students as water rolls off a duck's back. I should say, as water *used* to roll off a duck's back. Chemists recently discovered a substance, Aerosol OT, which breaks up the surface tension of water molecules. When a duck is shampooed with this chemical, he loses his floating power. With piteous loss of dignity, he sinks. This is because Aerosol removes the protective oils from the feathers, allowing water to penetrate. What we need is an Aerosol which, when poured over students, breaks down their resistance and allows ideas to penetrate.

"The human mind," says James R. Mursell, in *The Atlantic* for March 1939, "is not naturally docile. It is capable of remarkable feats of resistance and rejection, beneath a tame and dutiful exterior. It . . . makes its own only those things which, for some genuine reason, seem to matter."

Research in education at Columbia's Teachers College has supplied Professor Mursell with experimental data on numerous subjects—mathematics, for example.¹ Tests in this field carried out among students at various levels reveal a disconcerting fact. Students easily learn the routine operations of addition, subtraction,

multiplication, and division, and they can even solve equations of familiar types; but once they are faced with new situations—yet situations which require only such mathematical tools as they have demonstrated that they can use in standardized situations—they founder. In one test on 1200 students, five-sixths were successful *when told what operations to perform*, but only one-quarter succeeded on simple but *original* problems.

What would you think of a carpenter who, hired to build a bookcase, had to be told when to use a chisel and when to use a plane! That, in effect, is what must be done with a majority of students of arithmetic, algebra, and geometry. This is true whether training ceases with grade school or continues through high school and college. In fact, in some tests the major part of college freshmen came off worse than the high school students.

Similar difficulties are found in the teaching of the natural sciences, grammar, foreign languages (Latin in particular), and English. Not only do students have difficulty in retaining bare facts, but the skills acquired in learning one subject seem to help scarcely at all in learning another. There is distressingly little transfer of training. For example, the techniques which we assume the student learns in arithmetic and algebra seem to have vanished by the time he finds practical use for those techniques in physics or in chemistry. The student learns what

¹ See editorial page 181, *THE MATHEMATICS TEACHER*, April, 1939

the tools are, but he does not learn when and how to use them.

Sometimes it seems as though students are just plain "cussed." They seem to encase their minds in shells of purest ivory. Is that because they are determined not to learn? Or is it because the ideas themselves, or the settings in which the ideas are presented, are remote and abstract?

Consider, for instance, interest rate problems drawn from commerce and banking. How many junior high or high school students can identify themselves with the austere individual they think of as a banker? How many can mentally place themselves before a large, flat-topped desk, passing on bills of exchange, commercial paper, and corporation bonds? Yet the principles of interest demonstrated by banking problems are equally well demonstrated by problems in fields of personal experience.

If we have reason to assume that banking problems bore students, on what grounds do we assume that consumer credit problems bring them to life? The first answer is that such personal problems touch a very tender spot, the pocketbook. A majority of students live in families which are buying on the installment plan or making payments on money loans; and not a few students are themselves buying bicycles, wrist watches, or even second-hand automobiles on time payment plans.

The used car, let us say, costs \$130 on time, or \$100 in hard cash. Anyone using fourth grade arithmetic can see that the difference is \$30. To save even half that sum would take many trips to the cash-and-carry stores, a lot of comparisons on sheets or towels, and much scrimping on lunches. Because installment credit is so often bought in big amounts, careful shopping may yield relatively large rewards.

Some shoppers, upon learning the risks and the costs, husband their resources and pay cash. Many others say, "I must buy at once, but I can't lay my hands on \$100. For me, it is \$130 or no car!"

But is it? Must the credit purchaser accept the seller's terms or go without? Is there no other way to get \$100? As a matter of fact, scores of credit grantors are eager to accommodate consumers, with differing plans and differing prices.

The very fact that there is variety makes for confusion. There are, in the first place, other sellers on credit. By stretching charge accounts and postponing payment of bills, or by purchasing something else on time, the buyer often can substitute low-priced credit for high-priced. In the second place, the purchaser can borrow money and then, cash in hand, he can face the seller in a much improved bargaining position. Many banks now make small loans for the specific purpose of financing purchases, as do credit unions, industrial banks, and personal finance companies. The loan sources differ in many respects besides the price. There are differences in convenience and speed, required security, privacy of dealings, liberality of terms, strictness with which prompt payment is enforced, penalties for irregularity, and so on. Not a few borrowers, experience shows, are more interested in liberal service than in low cost. As with blankets or linoleum or permanent waves, the purchaser finds that he does not always get what he wants at the lowest price. In the final reckoning, however, all the service features of a credit or loan plan must be weighed in the balance against the cost.

For a reliable comparison of credit costs, we need an accurate gauge. Here per cent comes in handy. It is the one yardstick that reduces the three factors of principal and time and interest to a single common measure.

To be sure, we can justify the labor of calculating percentage rates only if costs vary widely. That they do, in fact, vary widely is shown by several studies. The one with the broadest coverage was made in 1936 by a Special Commission of the Massachusetts legislature. It revealed a chaotic lack of uniformity. Here are some

illustrative figures: On new car sales, with original unpaid balances of \$400 to \$500, the rates, inclusive of required insurance, ranged from 16% to 32% per annum. On used car sales with balances of \$100 or less, the rates were from 0% to 456%. On refrigerators, rates varied from 9% to 58%; on radios, from 0% to 881%; on furniture, from 11% to 330%; on vacuum cleaners from 8% to 73%.

Scanning the details of the listed cases, one soon discovers that it is impossible to make even approximate comparisons on the basis of dollars and cents charges. There are, for example, two furniture cases, each with a \$70 unpaid balance at the start. In one case the credit charge is \$22.50; in the other, \$10. But because of the difference in the length of terms, the \$22.50 charge results in a rate of only 16%, whereas the \$10 charge results in a rate of 36%.

If dollar costs of credit are deceptive, why not rely on the percentages quoted by the sellers? The Massachusetts Survey provides the answer. The rate most frequently quoted by the seller was 6%. Of the 500 cases, there were 105 in which dealers *quoted* that rate; yet, figured on the average amount of credit received, 6% was the *true* rate in only one case. Cost rates for the other 104 cases ranged from 7% to 679%. In a little more than one-half the 6% cases, the actual rates were between 11% and 20%. In eight cases the rates exceeded 100%. If those who buy things on "easy terms" want to know exactly how easy the charges are, they had better do the figuring themselves.

Rates on money loans also vary widely among different types of lenders and also, to some extent, among the lenders of any one type. Generally, the better the security, the lower the charge. Has the borrower any sort of bankable collateral—a savings bank account, building and loan shares, listed stocks or bonds, or a life insurance policy? Borrowing on these valuable assets, he seldom needs to pay more than a true 6%. But if the borrower

is like the great majority of hard-pressed families, he will not have access to assets adequate for the purpose.

Possibly the borrower is able and willing to provide endorsers for his note, to share responsibility of payment. If so, and he wants \$100 or more, he may be able to borrow from the personal loan department of a commercial bank. Frequently the charge is 5% or 6% discounted from the face of the note, equivalent to about 10% or 12% on the average principal. The range of rates is from 7% to about 23%. Morris Plan or industrial banks also lend on endorsed notes, and occasionally without endorsements, at rates, including fees, generally between 17% and 20% per annum on the average principal.

Co-operatives provide another source of comparatively low cost loans. Credit unions, as a rule, lend upwards of \$50 on notes endorsed by fellow members, and smaller amounts simply on the security of the borrower's credit union shares. By far the commonest rate is 12% a year, although some unions have rates as low as 6% and a few as high as 18%.

Borrowers sometimes have access to remedial loan societies. In the twenty-one cities in which they operate, they lend on various forms of security: pledged personal property, chattel mortgages on furniture and automobiles, and wage assignments. Rates, somewhat lower than for competing commercial agencies, range from a low of 9% a year to a high of 40% on the smallest and least secure loans.

The largest cash lenders are the personal finance companies. Regulated and supervised under special state laws, they serve a large body of low and moderate income families who, for one reason or another, cannot obtain loans at lower cost. The bulk of loans are secured by household furniture; others by automobiles, wage assignments and occasionally simply by the signatures of husband and wife. Maximum legal rates in most states are from 2½% to 3½% a month on unpaid bal-

ances, equivalent to 30% to 42% per annum simple interest. The average rates actually charged are considerably lower than that; and even lower rates often are obtainable on the larger-sized loans.

Lastly, of the states without operative regulatory laws, about twenty in all, the majority are overridden with illegal lenders, who charge what the traffic will bear. Seldom is the rate less than 60% a year. A common charge is 10% for a half month (240% a year), and rates as high as 1000% a year are not unknown.²

Through this wide variety of services and charges, retail credit and cash loans are available to the vast majority of families. Certainly credit users are interested in the prices they are asked to pay. At present, however, they are merely confused by the mumble-jumble of percentages, discounts, dollar charges, and fees. Without the help which mathematics provides, they are powerless to make accurate comparisons.

The smaller the amount and the shorter the terms, the larger is the per-dollar expense of granting the credit. On automobile tires, for example, a common charge is 10% of the unpaid balance for ten weekly payments. This is equivalent to about 95% per annum. Now, looking only at the seller's expenses, that rate may not be excessive; that is, the charge may just about cover costs. But it does not follow that the price is right from the consumer's standpoint.

Suppose you order an ice cream cone at the corner store, only to find that the price is 15¢; and suppose you believe the proprietor when he explains that his overhead

and salaries make that price necessary. You might say, "Yes, on the basis of your costs, that ice cream cone may be worth 15¢, but not to me. And I won't pay it!"

The user of consumer credit must likewise make up his mind on the basis of value received. He must not conclude that just because the credit charge is only a dollar or two, the price is immaterial. If the rate is in the neighborhood of 100%, he is buying a 25 cent ice cream cone.

Suppose, on the other hand, the proprietor tells you that the price is "reasonable" or "moderate" or "nominal," and that on further questioning he quotes the price as "6%." You wonder, "6% of what?" You cannot tell. All you know is that you "don't know nothin'." Then suppose the proprietor says that you may pay for the ice cream cone at the rate of one cent a week. Knowing that your budget can stand one cent a week, you sign a contract, make a one cent down payment, and walk out with the goods.

Ridiculous? Yes, and equally so at times is the behavior of the credit user. Mr. Credituser needs a new refrigerator, would like a new refrigerator, and his wife is determined that he buy a new refrigerator. Can he pay for it? Mentally he pares and prunes until he has made room in his budget for payments of \$11.55 a month. Then he buys.

Now, it is highly important that Mr. Credituser fit payments to his budget. If he fails, he will lose the new refrigerator, the old one which he traded in, all the money he has paid, and in addition may find himself liable for a deficiency judgment. He becomes a hard-up victim of "easy payments."

But let us suppose that he is able to meet the payments. Does that mean that the purchase is wise? Perhaps it is; perhaps it is not. Mr. Credituser must bear in mind that when he buys on time he buys two things: merchandise and credit. If he gets a good price on the refrigerator, but pays too much for the credit, he can-

² For a comprehensive survey of the whole subject, see "Credit for Consumers," pamphlet published in 1939 by the Public Affairs Committee, 30 Rockefeller Plaza, New York City. For a discussion of illegal lenders, see "Loan Sharks and Their Victims," published in 1940 by the same Committee. For a condensed summary of the small loan laws of the United States and the rates charged by legal lenders, see Pamphlet 37, published by Pollak Foundation, Newton, Massachusetts, 1939. Uniform price of all these pamphlets is 10 cents.

not consider himself a smart shopper. Students, seldom liberally supplied with spending money, take pride in learning to become smart shoppers.

Another reason that students take to consumer credit problems is found in selling psychology. The experienced salesman tells you, quoting "Sure-Fire Sales Slogan No. 7," that "the prospect is three-quarters sold, once he takes the product in his hands." If he can be made to feel it, smell it, examine it closely, hold it to the light, or ride in it, he will take a lively interest. That is why solid geometry comes to life when students make their own paper models of cubes, cylinders, and polyhedrons. Similarly, students who can be persuaded to bring to class their own interest problems, from terms sampled in the stores or, still better, from credit problems within their families, are visibly stimulated. At once they sense the usefulness of mathematics. At any rate, extended trial of family credit problems in actual teaching has shown that students do like them. Here are some of the unsigned comments, written by Newton High School students at the year-end. "The best to my opinion has been the credit work this last part of the year." "I think more time should be spent on problems in consumer credit; this will be more valuable to us in later life when we will have to face the problem of buying." Occasionally a comment ran: "The other part that I didn't care for especially was the figuring of consumer credit." But about nine out of ten students expressed approval of consumer credit examples, and teachers throughout the country have reported student reactions much in the same vein.

Teachers today are hard-pressed to keep up with the clock. After they have corrected papers, held classes, tutored backward John and Mary, attended meetings, analyzed "college boards," and corrected more papers, there seems precious little time for major operations on course con-

tents. Naturally they ask, "Does the introduction of new materials mean a thorough revamping of course outlines?"

Not necessarily. To be sure, some schools are working out entirely new courses, adapted to the "under-endowed students" who are likely to go no further with their study of mathematics. These courses are devoted exclusively to practical, familiar problems of every-day living which demonstrate the homely uses of mathematical tools. Their introduction into school curricula is revolutionary.

What is occurring in most high schools, however, is evolution—an accelerating evolution, but hardly a revolution. Emphasis on pure theory and abstract principles gradually is giving way to increased stress on practical applications.

Many of the newer mathematics texts organize the problems in groups of real-life situations: furnishing the home, running the car, buying gas and electricity, paying taxes, buying on installments, and so on. Although in some cases the mathematical principles appear to wear elaborate masquerades, at least they are present at the fancy dress ball. In other cases, the texts start with the principles and quickly move for illustration to close-to-home problems. Either approach can be used to awaken interest. The student does not object to principles, once he sees that they help him to cope with his environment. This applies equally to a student who goes on to advanced mathematics and to a student who uses his knowledge to figure out whether, in choosing a home, it is better to buy or to rent. In any case, only as a student comes to appreciate the concrete uses to which mathematics can be put will he retain the facts and the acquired skills beyond graduation day.³

³ As an aid to students in making such uses of mathematics, the Pollak Foundation, Newton, Massachusetts, has published "One Hundred Problems in Consumer Credit." Price, 10 cents.

Vitalizing Geometry with Visual Aids

By RICHARD DRAKE

University of Minnesota, Minneapolis, Minnesota

and

DONOVAN JOHNSON

North High School, Sheboygan, Wisconsin

GEOMETRY more than any other subject is dependent upon visual aids. Every theorem and every exercise requires a diagram in a book, on the blackboard or in one's mind. Yet probably in no subject have teachers shown less progress in developing visual aids and improved visual techniques. Newer textbooks have increased the number of photographs of church windows, bridges and decorative patterns, but much more visual material is needed to bring geometry closer to everyday things. The mathematics exhibit at the world's fair in Chicago indicates how interesting mathematics can be made when it is presented in concrete terms rather than in the usual abstract manner.

A survey of the literature on visual aids in mathematics indicates that very few visual aids in geometry are available. However, individual teachers have collected and used material in mathematics classes by means of the following projects.

1. Make models of mathematical instruments such as an abacus, pantograph, sundial, slide rule, transit and angle mirror.
2. Collect or make models of geometric solids such as cones, cubes, prisms, pyramids, cylinders and polyhedrons.
3. Make models for theorems of solid geometry.
4. Construct model scenes with papier-mâché and strings to illustrate problems in indirect measurement.
5. Take snapshots of moving points of light that describe loci.
6. Perform experiments to illustrate loci problems.
7. Make stereopticon pictures by taking two pictures of objects with two cameras set three inches apart.
8. Take snapshots of geometric de-

signs, curves, symmetry and proportions in the community.

9. Make movies of carpenters, surveyors, aviators and others actually using geometric skills.
10. Draw cartoons to illustrate problems.
11. Make a series of charts with step by step drawings of geometric constructions.
12. Take a field trip to observe the uses of geometry in the community.
13. Make a series of slides to illustrate everyday uses of geometry.
14. Enlarge diagrams and pictures by proportional parts.
15. Use mathematical instruments to measure distances indirectly.
16. Perform experiments to discover laws or formulas for acceleration of gravity, curves of projectiles, simple machines, sound waves, reflection of light, laws of chance, etc.
17. Collect slogans, proverbs, quotations or jokes of a mathematical nature.
18. Collect or devise mathematical puzzles such as magic squares, crosswords and fallacies.
19. Make geometric Christmas tree decorations.
20. Write or present a mathematical play.
21. Write a poem on mathematics.
22. Make posters, bulletin board displays or booklets on the following:
 - a. Geometry in nature—butterflies, bees, spider webs, leaves, snowflakes, flowers and crystals.
 - b. Geometry in the home—geometric designs in rugs, tiles, lace, quilts, furniture, pottery, gems, flags, stamps, clothes, textiles and blankets.
 - c. Geometry in industry—bridges, aviation, architecture, advertisements, navigation, engineering, invention, research, automobiles and transportation.

- d. Geometry in art—geometric designs, surrealism, Indian art, divine proportion and paintings.
- e. Geometry in drawing—blueprints, maps, scale drawings and mechanical drawings.
- f. Optical illusions.
- g. Geometry in churches.
- h. Geometry in everyday life—circles, triangles, polygons, parallel lines and symmetry.
- i. Graphs.
- j. Vocations requiring a firm foundation in mathematics.
- k. Reasons for studying geometry.
- l. Development of our number system.
- m. Calculating machines.
- n. History of mathematics.
- o. Mathematics problems in other courses.
- p. Development of measurement.
- q. Short cuts in mathematics.
- r. The world without mathematics.
- s. Ye curiosity shoppe of numbers.

Teachers should constantly be on the lookout for methods and devices for motivating their pupils, but the need for creating interest is especially great in an abstract subject like mathematics. The principal value of the visual material used by mathematics teachers is the motivation that results. Other objectives that are given for using visual material are to:

1. Connect mathematics with practical life experiences.
2. Develop an appreciation of the power, beauty and worth of mathematics.
3. Develop cultural and aesthetic values.
4. Show effectively the relation between mathematics and other subjects.
5. Develop the proper attitude toward mathematics.
6. Develop a mathematics atmosphere in the classroom.
7. Provide for individual differences.
8. Aid in developing perspective and visualizing three dimensions.
9. Aid the grasping of fundamental concepts.
10. Offer vocational training and guidance.
11. Awaken a casual public.

Although the mathematics teachers who use visual aids state that learning is improved, no objective data was obtained to substantiate their belief. However, since efficient learning takes place when fairly accurate and vivid imagery accompanies the learning of verbal symbols, teachers of mathematics should make more use of visual aids than they have in the past.

In order to have an effective visual aids program, the following equipment and supplies are needed in a mathematics' classroom.

1. Surveying instruments:
Transit, level, sextant, hypsometer, plane table, alidade, angle mirror, tapes, ranging poles, arrows, magnetic compass, planimeter.
2. Drawing instruments:
Drawing board, T-square, triangles, protractors, compasses, pantograph and proportioned dividers.
3. Blackboards:
Spherical, cross section and rotating.
4. Blackboard equipment:
Protractor, compass, pointers, meter sticks and colored chalk.
5. Collections:
Charts, posters, graphs, maps, projects, proverbs, booklets, pictures and flash cards.
6. Models:
Solids, dissected cone, contour maps.
7. Mathematical instruments:
Calculating machine, demonstration slide rule and abacus.
8. Science apparatus to demonstrate formulas:
Seconds pendulum, levers, gears, balance, inclined acceleration plane and resolution of forces.
9. Visual aids equipment:
Radio, motion picture projector, reflectoscope, projection lantern, screen, lantern slides and photographs.
10. Museum collection of models of ancient mathematical instruments.
11. General equipment:
Bookcase, equipment case, museum case, filing cabinet, bulletin boards, display rack, books, magazines, wall pictures related to mathematics.

Much of this equipment can be used in cooperation with other departments.

In this classroom, mathematics can be taught as a science of concrete relationships. It here becomes alive and practical. Thus it meets the challenge of modern criticism.

There are numerous sources for visual aids in geometry. Some of them are listed below:

Pictures:

Tree of Knowledge—Museum of Science and Industry, Chicago, Illinois.
Murals at Lincoln School, Columbia University.

The History of Measurement—Laura Christman, 1217 Elmdale Ave., Chicago, Illinois.

The History of Optics—Bausch Lomb Co., Rochester, New York.

Old Persian Astrolabes—U. S. National Museum, Washington, D. C.

The Compass of George Washington—ibid.

Facsimile of Pages from Old Manuscripts—British Museum, London.

Oxford Science Cards—Oxford University Press, New York City.

Portraits of Mathematicians—Scripta Mathematica, Perry Picture Co., Open Court Publishing Co., University Prints Co., U. S. National Museum.

Mathematics of the Automobile—Chevrolet Motor Co.

Mathematical Themes in Design—Scripta Mathematica.

Lantern Slides:

Symmetry—American Museum of Natural History, New York City.

Indian Art—American Museum of Natural History, New York City.

History of Mathematics—Bureau of Publications, Columbia University.

Portraits of Mathematicians—Central Scientific Co., Chicago.

Motion Pictures:

Geometry—two reels, 35 mm, Wholesale Film Service, Inc., Boston.

The Play of Imagination in Geometry—one reel 16 mm, University of Wisconsin

Frequency Curves— $\frac{1}{2}$ reel, 16 mm, Eastman Kodak Co., Rochester, N.Y.

Einstein's Theory of Relativity—two reels, Institutional Cinema Service, Inc., N.Y.C.

The Engineer—one reel, 16 mm, Edited Pictures System, Inc., N.Y.C.

The Skilled Mechanic—one reel, 16 mm, ibid.

The Metric System—Bray Productions, N.Y.C.

Definition of Plane Geometry—two reels, Art Film Co., Boston.

Mysteries of Snow— $\frac{1}{2}$ reel, Bray Productions, Inc., N.Y.C.

Film Slides:

Fractions—Society for Visual Education, Chicago.

Geometry—Society for Visual Education, Chicago.

Stereopticon Pictures:

Unit of 25 Stereopticon Drawings of Solid Geometry, \$5.00—Keystone View Co., Meadville, Pennsylvania.

Three Dimensional Pictures:

Diagrams in Three Dimensions for Solid Geometry, \$2.00—The Orthovis Co., Chicago.

Excursions in Mathematics, \$1.20—ibid.

The purpose of this article has been to give a brief survey of the objectives and techniques of visual aids in teaching mathematics and to outline material which may be of help to the teacher in the field. No attempt has been made to prescribe a set course to be followed in instituting a program of visual education in mathematics. Rather, it is hoped that the material presented will serve both as an incentive to progressive teachers to investigate further what has been done in this field, and as a basis for the production of future projects.

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LONG AGO, the Dutch purchased Manhattan Island from the Indians for \$24.00 and a barrel of whiskey. A bargain for the Dutch, you may conclude. But not so!

A banking expert figures out that \$24.00 invested at compounded interest since the date of the purchase would by now have increased to a little more than \$4,000,000,000 whereas the Island was valued a year or two ago at only \$3,800,000,000. The purchasers, or their unlucky heirs, are about \$200,000,000 in the hole.

A dollar or fifty cents too much was paid for the Island. Not a large amount, to be sure, but cheating is cheating; and we feel that the Indians should give that amount back.—*The Mathemadison*.

Analytic Geometry in the High School

By ARTHUR F. LEARY

English High School, Boston, Massachusetts

WHAT little work on graphs is included in the usual mathematics course in the high school could readily be extended to include more of the elements of analytic geometry. Most pupils like graphs and find that problems requiring the field of coordinates are easier and more interesting than straight manipulation.

The new attainment test known as Mathematics Gamma, now being offered by the College Entrance Examination Board, is apt to contain questions in analytic geometry. With this in mind some teachers are making alterations in their preparatory course. During the past two years I have extended my teaching of graphs so as to include more of the elements of analytic geometry. In my experimenting I have found that the graphical work ties in not only with the algebra, but also with the trigonometry and geometry. I think that it is best not to teach the analytic geometry as a separate entity, but to introduce phases of it here and there throughout the year. If a pupil is told that he is now studying analytic geometry he may tighten up and feel that it must be hard.

The use of graph paper is introduced in grade eight or nine, where linear equations are drawn and intersections located. Here that handy word "locus" is also introduced. A more extended use of graphs customarily comes, however, during the second course in algebra. My thought is to extend this field so as to include not only the traditional discussion of the conic sections, but also some problems involving slope, distance, perpendicular lines, angle between two lines, center of a circle, and the more technical definitions of the conic sections.

This newer material, as I have been using it, comes in the senior mathematics course (grade twelve), which, in the

Boston English High School, comprises elementary algebra complete and trigonometry. Until recently this course pointed to the College Board examinations Mathematics A and E, while the course of our junior year (grade eleven) prepared for the examinations in geometry, Mathematics C and D. We are now altering the courses a little. In the third year we are putting less accent on formal demonstration, especially in solid geometry, and are inserting more algebra. In the fourth year we are adding the analytic geometry.

STRAIGHT LINES

At the time when simultaneous equations are being studied, I teach not only the graphing of straight lines to find the point of intersection, but go on to some of the newer work. I use the same kind of graph paper, 8"×10" with $\frac{1}{4}$ " squares. My first lesson is on finding the distance between two points. No new formula is necessary, just a hint that most problems on distance involve in some way the Pythagorean theorem. When I took this up last November, I gave for a home lesson two problems in which the vertices of a triangle were given, and the question was to find the lengths of the sides. Another problem gave the vertices $(-3, 2)$ $(6, 5)$ $(3, -1)$ of a triangle and asked if this triangle was isosceles. A fourth one was to show that the diagonals of a given rectangle were equal; and finally "if the vertices of a quadrilateral are $(0, 2)$ $(7, 1)$ $(12, 4)$ and $(5, 5)$, is this figure a parallelogram?"

The second lesson was to find the midpoint of a segment. This home lesson was: (1) The vertices of triangle ABC are respectively $(1, 1)$ $(6, 3)$ $(4, 7)$, what is the length of the median from vertex A ? (2) The points $(1, 2)$ $(8, 1)$ $(13, 4)$ and $(6, 5)$ are the vertices of a parallelogram;

find the length of each diagonal and find the coordinates of the midpoint of each diagonal. (3) The vertices of a triangle are $(2, 2)$ $(4, -6)$ $(8, 4)$; show that the line joining the midpoints of two sides is equal to one-half the third side. (4) The vertices of a quadrilateral $ABCD$ are $A(0, 0)$ $B(0, 4)$ $C(6, 6)$ $D(8, 0)$; prove that the lines joining the midpoints of the sides form a parallelogram.

Next comes the equation of a line through two points. At the time I took this up I had not yet discussed the tangent of an obtuse angle. For this reason I confined myself for the time being to positive slopes. The form $y = mx + b$ is the most useful straight line formula. It is readily explained by drawing a straight line through the origin; the slope of this line is obviously y/x . Then draw another line parallel to this one, and crossing the y axis at the point $(0, b)$. The slope (m) of this line is $(y-b)/x$, that is, $m = (y-b)/x$, or $y = mx + b$. With two points given, the slope of the line joining them can first be determined, and then by substituting the coordinates of one of the points for x and y in the equation $y = mx + b$, the value of b is ascertained. Example: What is the equation of the straight line through the points $(4, 3)$ and $(1, -2)$? After plotting these two points, (every pupil should make a graph in connection with every problem) the slope is noted as $5/3$. In the equation $y = 5/3x + b$, substitute the coordinates of one of the given points, say $(1, -2)$, then $-2 = 5/3 + b$, or $b = -11/3$. Then the desired equation is $y = 5/3x - 11/3$ or $5x - 3y = 11$.

Here are type exercises assigned at this time: (1) What is the equation of the line whose slope is $5/4$ and which goes through the point $(1, 4)$? (2) What is the equation of the line through the points $(-3, 1)$ and $(8, 6)$? (3) Draw a quadrilateral, designate the coordinates of the four vertices and also designate the coordinates of the midpoints of the sides; then prove that the line joining two adjacent midpoints is parallel to (has the same slope as) the line

joining the other two midpoints. (4) Draw the line $F = 9/5C + 32$. What is the centigrade equivalent of normal body temperature, 98.6°F ?

On the sixth day of the graphical work this year I gave the following test: (1) Solve graphically and check a pair of linear equations that I selected. (2) The vertices of a triangle are $A(-2, -2)$ $B(6, 0)$ $C(4, 4)$, the midpoints of AB , BC , and CA are, in order, D , E , and F ; (a) designate the coordinates of points D , E , and F ; (b) what are the lengths of the segments FE and AD ? (c) what are the slopes of the lines AC and DE ? (d) what is the equation of line AB ?

The results of this test were average; the first problem gave as much bother as the second. The pupils took this new work in stride and found no particular difficulties with it.

Further work on coordinates was then postponed until we studied the trigonometric functions of angles of all sizes, and the expression "negative slope" took on some meaning. The first assignment was to figure out the slope of each of a dozen lines whose equations were given in the algebra text; also to show that the line joining two given points, such as $(-3, 4)$ and $(6, 1)$, was parallel to the line joining two other points, such as $(7, 2)$ and $(5, 8/3)$. Here is a test I gave at this time: (1) Given, line $3x - 4y = 6$, (a) plot the line; (b) what are its intercepts? (c) what is its slope? (d) at what angle does it meet the x axis? (use tables); (e) what is the equation of the line through the origin and parallel to the given line? (2) (a) What is the equation of a line whose slope is -2 and which passes through the point $(-1, 6)$? (b) What is the equation of a line whose intercepts are $(3, 0)$ and $(0, -5)$?

Problems which ask for the relation between two variables when sets of values are given can now be done. Consider this problem: State the relation between a and b if $a = 1$, when $b = 1$, $a = 2$ when $b = 5$, $a = 3$ when $b = 9$, $a = 4$ when $b = 13$. These

points plot on a straight line. Choosing any two of the points we find the equation to be $b = 4a - 3$.

This lesson was followed by a discussion of perpendicular lines. The very brief derivation of the "negative reciprocal" formula tied in nicely with the trigonometry we had just been studying, namely $\tan(90^\circ + \theta) = -\cot \theta = -1/\tan \theta$. The problems under this heading went very well. They included the proof that the line joining two certain points was perpendicular to the line joining two others, and that a given triangle was right-angled. Another type was to find the equation of a line through a given point $(5, 8)$ perpendicular to a given line, $3x + 7y = 21$. The next lesson included both parallel and perpendicular lines. One problem in this home-lesson was as follows: What is the equation of the locus of points equidistant from $(-2, 5)$

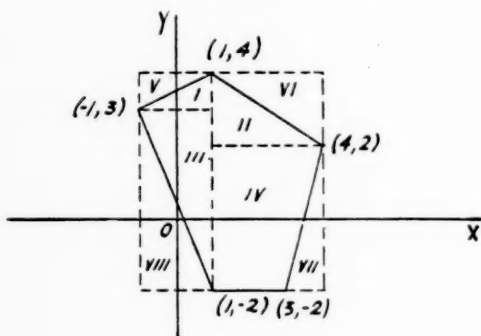


FIG. 1

and $(4, -3)$? Designate the coordinates of some point on this locus and show that this point is equidistant from the two given points. Another problem: If the coordinates of three of the vertices of a rectangle are $(2, 1)$, $(5, -2)$ and $(10, 3)$, find the coordinates of the fourth vertex.

This set of lessons also included problems on areas, in which lines parallel to the axes could be so drawn as to form right triangles, rectangles and trapezoids. As this method is used in topographical work an example will be given. What is the area of the polygon shown in figure one? The pentagon can be split up into three right

triangles I, II, and III, and trapezoid IV, whose areas are respectively 1, 3, 5, and 10 square units to give a total area of 19 square units. Likewise, a rectangle can be circumscribed about the polygon, with a gross area of 30 square units. From this subtract the areas of right triangles V, VI, VII, VIII, i.e., 1, 3, 2, and 5 for a total of 11 square units, to give the net area of the given polygon as 19 square units.

As an optional problem for the brighter pupils I asked them to find the distance from the point $(3, 5)$ to the line $2x - 3y + 6 = 0$. This exercise did not go very well, but one boy turned in a neat and simple solution involving only corresponding sides of similar triangles, as follows: The line $x = 3$ crosses the given line at the point $(3, 4)$. Then we get the proportion

$$\frac{d}{1} = \frac{6}{\sqrt{52}} \quad (\text{see figure two}).$$

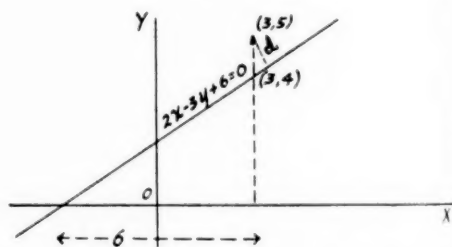


FIG. 2

So far the number of lessons devoted to the new work is about twelve. At this point there is a lapse in the study of the analytic geometry; that is, I go on with the regular instruction as given in former years until such time as I take up the formula for the tangent of the difference of two angles. More of this new work can then be inserted, namely problems on the angle between two lines. This type of problem helps the pupil to associate graphical work with trigonometry and to dispel the notion that it is a part of algebra only.

The formula is readily derived. In figure three $\phi = \theta_1 + (180^\circ - \theta_2)$, so $\tan \phi = \tan[\theta_1 + (180^\circ - \theta_2)]$ or $\tan \phi = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$.

With this formula the angles of a triangle can be found when the coordinates of the vertices are given. Or the following prob-

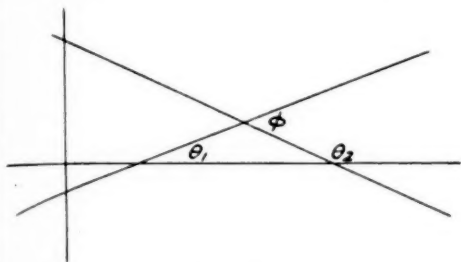


FIG. 3

lem can be solved: Find the slope of a line which makes an angle of 30° with a line whose slope is 3.

These elementary notions on the analytic geometry of the straight line can then be taught in about thirteen or fourteen lessons. They can be grouped as follows:

First group, to be studied at the same time as simultaneous equations and to include (a) distance between two points, (b) coordinates of the midpoints of a line segment, (c) equation of a line between two points, using only positive-sloped lines.

Second group, to follow the study of the tangent of an obtuse angle and to include (d) the slope of a line and the angle at which a line meets the x axis, (e) equation of a line of positive or negative slope, (f) perpendicular lines, (g) areas, (h) distance from a point to a line.

Third part, to be taken up when the required trigonometry has been studied, (i) angle between two lines.

Many problems in analytic geometry require algebra in their solution. It is just as well for high school pupils to realize that algebra, like arithmetic, is a tool. The study of algebra is not an end in itself; it has its application in other fields, such as this coordinate work. Likewise a tie-up with geometry can be brought out by assigning certain theorems of plane geometry to be proved by analytic methods.

CONIC SECTIONS

At the time when the regular graphical

work on the circle, parabola and other curves is being discussed, the new matter should be inserted, supplementing and extending the customary treatment of these curves.

Circle. Yes, a circle can have its center at some point other than the origin! But to assign without explanation the graphing of a circle such as $x^2 - 6x + y^2 + 4y - 23 = 0$ would be the wrong method of attack. Here any assumed value for y calls for the solution of a quadratic equation to find the corresponding values of x . The labor involved in making a key is so great that some method of simplification should be sought. Thus the analysis which leads to an easier way of graphing this curve is seen to be to a definite purpose.

This analysis is a simple application of the Pythagorean Theorem. What is the equation of a circle whose radius is r and whose center is the point (a, b) ? For any point (x, y) on the curve a right triangle can be drawn whose legs are either $(x-a)$ or $(a-x)$ and $(y-b)$ or $(b-y)$; in either case we get the relation $(x-a)^2 + (y-b)^2 = r^2$.

The first home lesson involving circles, as I teach them, comprises four graphs. The first two graphs are something in review—straight lines and graphs of quadratic equations. The other two problems I assign are circles, center at the origin, and cut by a straight line—to find the points of intersection. The pupils make keys, using radicals and numerous plus-or-minus signs. The second lesson is typified by these two examples: (1) What is the equation of the circle whose radius is $4\frac{1}{2}$ and whose center is at $(6, -1)$? (2) What is the equation of the circle that has for a diameter the line joining the points $(3, 4)$ and $(-2, 0)$? A third lesson has problems of this type: (1) In the circle $x^2 + y^2 + 8x - 6y - 11 = 0$, what is the radius and what are the coordinates of the center? (2) Likewise for $x^2 + y^2 + 6y - 16 = 0$.

This new work on the circle requires three lessons.

Ellipse. Mathematics, as taught in col-

lege, is highly abstract; applications are avoided. This technique is in keeping with the nature of pure mathematics; but it causes students to look on the study as dry and deadening. In high school I think we try to make the subject interesting, and frequently use the concrete-to-abstract procedure so as to help motivation. The ellipse, for instance, can be introduced by describing its use in bridge design, photographs of beautiful spans being shown to the class. Parenthetically, a future development in the teaching of mathematics consists in a greater use of pictures, not only on the class-room bulletin boards, but projected on a class-room screen. Again, the fact that the earth and the other planets travel in elliptical orbits with the sun at one focus is always of interest.

In my classes I do not deal with ellipses other than those symmetrical about the x and y axes, with center at the origin. The curve is first drawn on the blackboard, one pupil holding a piece of string against the board in two places (foci), allowing some slack. Another then stretches the string taut with a piece of chalk and thus traces the ellipse. The intercepts are then measured, after which the equation of the ellipse may be written as $x^2/a^2 + y^2/b^2 = 1$ where a and b are the x and y intercepts as actually measured. This equation can be put in the form $b^2x^2 + a^2y^2 = a^2b^2$. Using this equation it can be pointed out that the distinguishing characteristic of the equations of the circle and the ellipse is that in the circle a and b are the same while in the ellipse they are different.

I do not require that the pupils learn the derivation of the general formula of the ellipse. We cannot be as rigid in high school as in college. We should frequently point out how the mathematics under discussion can be used. It is as important to teach the pupils in high school that mathematics is a live subject as it is to insist on absolute rigidity at every step.

An ellipse is the locus of a point which moves in a plane so that the sum of its

distances from two fixed points (foci) is constant. From the method used in drawing the ellipse on the blackboard it is obvious that the curve drawn satisfies this definition.

To plot an ellipse from a given equation no elaborate key is necessary—just the four intercepts and one additional set of four points is usually sufficient. Any home lessons I assign involving the ellipse are of a routine type—plotting curves and finding intersections. The pupils could be asked to find somewhere the equation of the earth's orbit. Hogben, in *Mathematics for the Million*, gives the following equation of the moon's orbit about the earth, considering a point about 11,000 miles from the earth as a focus: $x^2/(238,833)^2 + y^2/(238,470)^2 = 1$. Here the eccentricity is only 0.055 and the orbit is very nearly a circle. Again, the ellipse has a property of reflection that is used in the design of whispering galleries. Focal radii to any point on the curve make equal angles with the normal to the curve at that point. Hence if the ellipse served as a reflector, a ray of light or of sound emitted from one focus would be reflected to the other.

Parabola. The parabola is already familiar to the students through its study in the opening lessons on the quadratic equation. Here the form of the equation most commonly met was $ax^2 + bx + c = y$. Such an equation with the variables exchanged gives a like curve turned sideways. Thus the equation $x^2 - 3x + 2 = y$ plots with the axis of symmetry vertical, while in the curve $y^2 - 3y + 2 = x$ it is horizontal. For purposes of analysis the curve is commonly drawn in this second position.

In my lessons on the parabola I include problems on the axis of symmetry and the lowest point of a curve. What is the axis of symmetry of the curve $y = x^2 - 3x - 4$? The curve crosses the x axis at $(-1, 0)$ and $(4, 0)$; halfway between is $(3/2, 0)$ and so the axis of symmetry is the straight line $x = 3/2$. Again consider the equation $y = x^2 + 3x + 5$. What is the lowest point on

this curve? Here the y intercept is $(0, 5)$. When $y=5$, $x=0$, or $x=-3$. The axis of symmetry then is $x=-3/2$ and the lowest point on the curve is $(-3/2, 11/4)$.

While I say little in my classes about eccentricity and asymptotes, I do require the pupils to learn the definitions of the conic sections and also to know how these sections can be cut on a right circular cone.

instructive and colorful model. Another one is made of strings and beads, the strings are elements and are sufficiently numerous to represent a cone; the colored beads are so placed on the strings as to outline an ellipse, a parabola, and a hyperbola.

A parabola is the locus of a point that moves in a plane so that its distance from a fixed point (focus) equals its perpendicular distance to a fixed line (directrix).

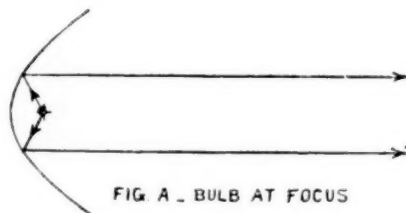


FIG. A - BULB AT FOCUS

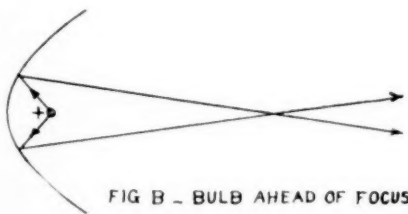


FIG. B - BULB AHEAD OF FOCUS

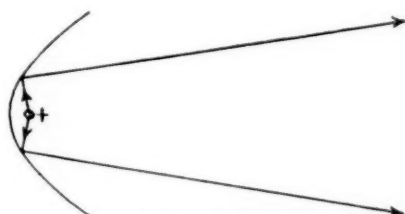


FIG. C - BULB BEHIND FOCUS

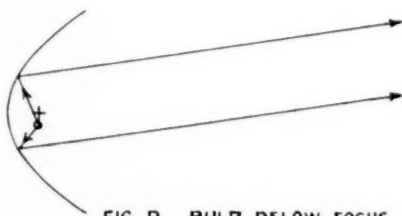


FIG. D - BULB BELOW FOCUS

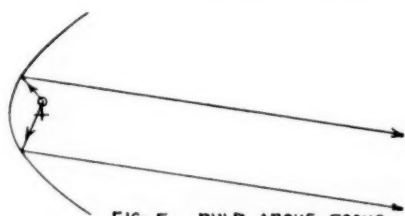


FIG. E - BULB ABOVE FOCUS

Figures D and E show the operation of the two filament tilting-beam bulb. When the switch is thrown from the lower filament to upper, the beam tilts from high to low.

The teacher should have a wooden model of a right circular cone, preferably of two nappes, with sections cut so as to illustrate the ellipse, parabola and hyperbola. By the use of dowels such a model can be taken apart and re-assembled at will. Some pupils are very good at making models. At my school we have on exhibition in a glass case a collection of well-made models. One is of celluloid, and of generous size. Sections in colored celluloid are inserted in the hollow cone to make an

lar distance to a fixed line (directrix). On account of the limited time available I do not attempt much in the way of derivation of the typical equations, or of the proof of the property of reflection.

There are several interesting applications of the parabola.

(1) The property of reflection—if the parabola is a reflector, a ray of light from the focus strikes the reflector so as to make the angle of reflection equal to the angle of incidence and is reflected along a line

parallel to the axis of the parabola. Thus a concave reflecting surface, such as an automobile headlight, in the form of a surface generated by revolving a parabola about its axis (paraboloid) would reflect all rays from a source at the focus in lines parallel to the axis of the reflector. Figure A illustrates this condition. The other four figures show what happens when the source of light is not exactly at the focus. (Note: these illustrations are from a circular sent to testing stations by the Registrar of Motor Vehicles of Massachusetts).

(2) A bridge may be designed as a parabola as well as an ellipse. The curve $x^2 = 2py$ gives a parabolic arch when p is negative. Without much explanation pupils can handle a question of this type: What is the equation of a parabolic arch of 60 feet span, and 20 feet high at the center? Compute the heights at intervals of five feet from the center.

(3) A suspended cable, uniformly weighted, such as on Brooklyn Bridge, takes the shape of a parabola.

(4) The path of a projectile is practically a parabola. This fact opens up the field of ballistics; at least, any pupil who is interested in military matters must appreciate that a knowledge of the parabola is necessary in the study of gunnery. Problems are available in which the pupil plots horizontal distance against vertical height attained, with the time in seconds shown on the same graph.

(5) Graphical problems can be made up involving the familiar formulas $v^2 = 2gs$ and $s = \frac{1}{2}gt^2$.

Hyperbola. A hyperbola is the locus of a point that moves in a plane so that the difference between its distances from two fixed points (foci) is constant. In teaching, it should be associated with the ellipse, as the form of equation is so similar. In the equation $x^2/a^2 - y^2/b^2 = 1$, a is the x axis intercept, while b is such that one focus is at the point $(\sqrt{a^2 + b^2}, 0)$. That is, the slope of one asymptote is b/a . When the asymptotes cross at 90° , the curve is called an equilateral or rectangular hyperbola

and its equation has the general form $x^2 - y^2 = a^2$; but when the axes are rotated through -45° the equation becomes $xy = a^2/2$. This tells us that if the product of two variables is constant, the curve is an equilateral hyperbola. Thus, if a rectangle is to have a given area, such as 144 square feet, and a graph is made of suitable sets of dimensions, (1×144) , (2×72) , (6×24) etc., the resulting curve is an equilateral hyperbola.

An interesting characteristic of the hyperbola is that the focal radii to any point of the curve make equal angles with the tangent at that point.

Tangents. I take up problems involving tangents late in the year with a special group comprising pupils who have a definite objective, such as examinations for college entrance or for some special scholarship. As regards a tangent to a circle, the slope of the tangent is the negative reciprocal of the slope of the radius drawn to the point of contact. Example: What is the equation of the tangent to the circle $x^2 + y^2 - 14x - 4y - 5 = 0$ at the point $(10, 9)$? This equation can be put in the form $(x-7)^2 + (y-2)^2 = 58$. The slope of the radius drawn to the point $(10, 9)$ is $7/3$. Then the slope of a normal to this line is $-3/7$. The equation of a tangent with this slope is $y = -3/7x + b$. For the point $(10, 9)$, $b = 93/7$, so that the desired equation is $3x + 7y = 93$.

The following is a problem involving a tangent to a parabola: Find the value of k such that the straight line $x - 2y = k$ will be tangent to the curve $y^2 = 4x$. A rough graph shows that k is negative. Solving the two given equations simultaneously by eliminating x , we get the quadratic equation $y^2 - 8y - 4k = 0$; from this $y = 4 \pm 2\sqrt{4+k}$, these two values of y being the ordinates of the points of intersection of the straight line and the parabola. For tangency these ordinates are equal, and so $k = -4$.

Another problem of this type: What is the equation of the straight line through

the origin and tangent to the curve $4y = x^2 - 2x + 9$?

I do not teach the standard formulas for the tangent to a curve at a given point.

OTHER CURVES

Graphs are an aid in the teaching of trigonometry. How does a pupil learn the fact that no real angle has a sine greater than one? First, from the definition he learns that the sine ratio cannot give a quotient greater than one; then from a perusal of a sine table he finds no values greater than one. Finally, a good look at the sine curve makes a visual impression which completes the lesson. I wonder how it would be to reverse this order and show the curve first, explaining it all later?

Here are a few valuable exercises for home lessons:

(1) Draw the cosine and secant curves on the same field, say for angles from $-\pi/2$ to $3\pi/2$. Take off the ordinates for any one angle, say somewhere between $\pi/3$ and $\pi/2$, and multiply these values. What is the product? Do this for three other angles.

(2) Draw the tangent and cotangent curves on one field and at some such ordinate as 0.4 take off the corresponding abscissas of the two curves. What is the relation between these two angles? Try this for two other ordinates.

(3) Plot the curve $y = \log x$ for values of x from 1 to 10, using a sheet of graph paper ten inches long and making the y scale 10" to 1 unit. Between each two consecutive points used in plotting the curve draw the x and y increments. How do all the x increments compare? The y increments? What is the tabular difference between $\log 2.5$ and $\log 3.5$? between $\log 7.5$ and $\log 8.5$?

(4) Plot the curve $y = 2^x$ for values of x from -2 to $+5$. Find from the graph the values of $2^{2.5}$ and of $2^{3.5}$? Multiply these two values.

The sine and cosine curves are met in the study of wave motion, alternating currents, and the measurements of the spec-

trum. Again, graphs of the trigonometric functions illustrate the continuity of angles, positive and negative.

When I am teaching coordinate geometry I make it a point to mention solid analytic geometry, so that the pupils will at least know that there is such a field. They readily guess the equation of a sphere, $x^2 + y^2 + z^2 = r^2$, but the question as to what figure is represented by the equation $ax + by + cz = d$ always brings the answer "a straight line." In such a question as, "What do you think the equation of a plane would be?" the response is not very good. At this point I emphasize the relation between the equations for corresponding figures of plane and solid geometry. Just as the sphere equation is formed from the circle equation by the addition of the z^2 term, so also, a line goes into a plane by adding a " z " term. Illustrations: $2x + 3y = 5$ represents a straight line on a two-dimensional graph, while $2x + 3y + 4z = 5$ represents a plane in solid analytic geometry. I am content to discuss briefly the form of the equation of a plane and the fact that one way of representing a straight line in equation form is as the intersection of two planes.

It is a distinct benefit for any teacher of mathematics to have available a black-board space marked off in squares. Such a board will be found convenient not only in teaching graphical work but also in explaining constructions, locus problems, and even in verbal problems in algebra.

The total number of lessons required for the graphical work discussed above is about twenty, divided somewhat as follows—straight lines, 13 days; circles, 3 days; the other curves, 4 days. All this is over and above the time allotted to the customary graphical work of the regular course. Some of the problems should be assigned only to the bright pupils of the class. Additional time can be given late in the year to a special group meeting after school.

It is obvious that in order to introduce new material in a course that already re-

quires all the available time, some of the old matter must be dropped. In my actual teaching during the past two years I have been able to save very little time from algebra. This past year I omitted synthetic division, and I assigned problems on the square root of a binomial surd and a few other such techniques to the brighter pupils only. In the trigonometry, this year I omitted the law of tangents and the formula $\tan \frac{1}{2}A = r/(s-a)$; and I took up the $\sin A + \sin B$ group of formulas with those only who were preparing for examinations. At the end of the year I had no time for a general review except insofar as the solution of verbal problems and of trigonometric equations involved review. I was able to save some time in the course in solid geometry, due to the lessening of

the emphasis on formal demonstration; but in my school, solid geometry is taught in the junior year, together with plane geometry. Our present plan is to insert more algebra into this junior year course and thereby save a little time for the analytic geometry in the senior course.

In my classes I have done nothing at all with calculus. As regards advanced algebra, I find that some of the smarter pupils can figure out in their own way problems that are ordinarily considered as advanced algebra. I give no formal instruction in the subject, but I do give formulas or other information to any pupils who make inquiry. The coordinate geometry, however, I am now inserting into the fourth year course, and the pupils seem to like it.

LOGIC IN GEOMETRY

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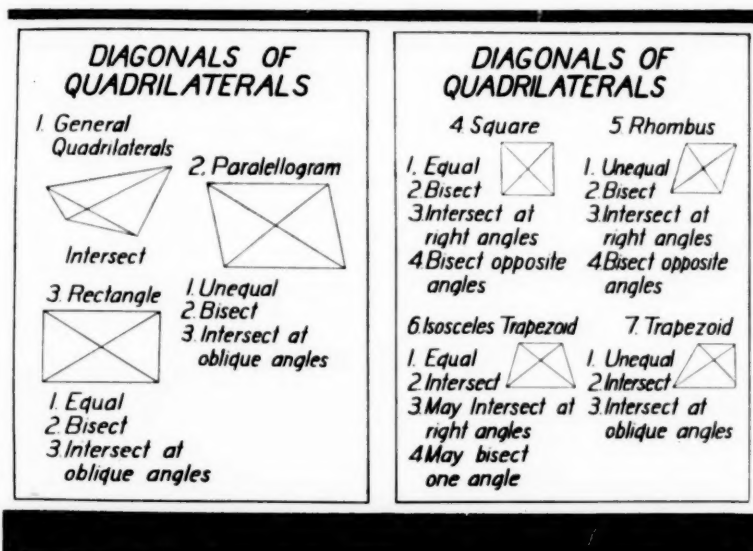
By HALE PICKETT
Ohio University, Athens, Ohio

GEOMETRY has been questioned as a high school subject by the general educator. However, this criticism has not been due so much to the subject as to the manner of teaching.

Only, within the last decade have specific procedures been given for the teaching of geometry. Recent literature indicates the necessity for using the function concept when teaching demonstrative geometry but the textbooks and professionalized subject matter offer only meager sugges-

tionships between changing variables. As has been previously stated: "to be is to be related and to know is to see relationships."

A static relationship is generally easy to observe but a dynamic relationship is more engaging and also more difficult to observe. Although the idea of *change*, involves difficulties, we must adjust ourselves to these difficulties, since this is a changing, or a dynamic world. As has been tritely stated: "Time marches on." What I am trying to emphasize is:



tions relative to practical classroom procedures.

Therefore, I am challenged to organize practical units involving functional thinking in Plane Geometry. Before presenting such a unit, a few general statements supporting the functional concept may be stimulating.

One's success in mathematics will depend on his ability to see relationships. In fact, success in life depends on one's ability to observe relationships in social trends. In mathematical terms, success is a function of the individual's ability to see rela-

that functional thinking in geometry approaches the type of thinking required in real life.

I want the pupils trained to see things change together. Functional thinking in geometry is sometimes described as the geometry of motion. What corresponding changes occur in geometric figures, when certain lines are lengthened, angles decreased, and points moved to infinity? In 500 B.C. Heraclitus, a Greek philosopher, pointed out that *change* was the only permanent factor in life.

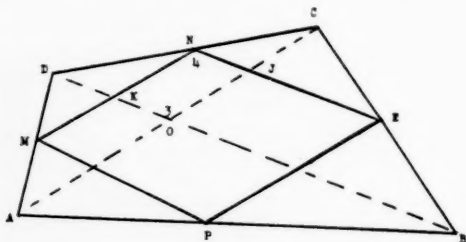
As a general aim, we shall accept the

theory of transfer wherein identical elements are involved. This is true in the geometry of motion; since all the parts remain the same except the changing variables. Therefore, functional thinking in geometry presents ideal conditions for transfer.

After having completed a discussion relative to the diagonals of quadrilaterals, the following facts are summarized relative to the diagonals of quadrilaterals.

These relationships relative to the diagonals of various quadrilaterals shall be utilized in a unit involving functional thinking.

Taking the case of a general quadrilateral and joining the midpoints of its sides, we have a general parallelogram.



Given: $P, E, N,$ and M midpoints of quadrilateral $ABCD$

To Prove: $PENM$ a parallelogram

Proof: Draw auxiliary lines CA and BD

$$MN = \frac{1}{2}AC \text{ and } PE = \frac{1}{2}AC$$

$$MN \parallel AC \text{ and } PE \parallel AC$$

$$MN = PE \text{ and } MN \parallel PE$$

$\therefore PENM$ is a parallelogram

Then, $KOJN$ is a parallelogram

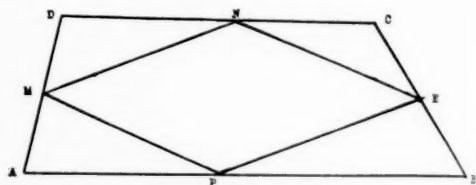
$$\angle 3 = \angle 4$$

Then we may state the theorem: *In any quadrilateral, the new quadrilateral formed by joining the midpoints of its sides is always a parallelogram whose sides and angles are functions of the lengths of the diagonals and the angles formed by the intersections of the diagonals of the original quadrilateral.*

CASE I

Given: Trapezoid $ABCD$ with $P, E, N,$ and M midpoints of its sides.

To determine the kind of a parallelogram formed by joining these points.



Since the diagonals of a trapezoid are unequal the adjacent sides of the parallelogram will be unequal.

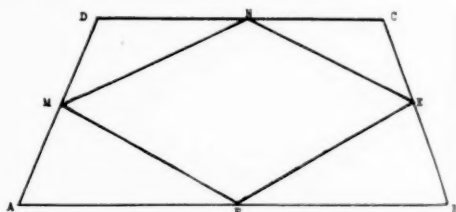
Also since the diagonals intersect at oblique angles the angles of the parallelogram will be unequal.

Therefore, the parallelogram $PENM$ will be a general parallelogram.

CASE II

Given: Isosceles trapezoid $ABCD$; with $P, E, N,$ and M midpoints of its sides.

To determine the kind of parallelogram formed by joining these points.



Since the diagonals of an isosceles trapezoid are equal the adjacent sides of the parallelogram will be equal.

Since the diagonals of isosceles trapezoid may intersect at *oblique angles* or at *right angles* the parallelogram $PENM$ will be a *rhombus* or a *square* (a special case of the rhombus).

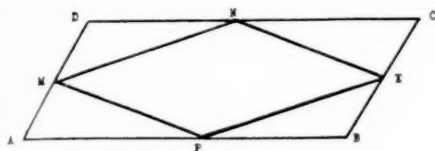
CASE III

Given: General parallelogram $ABCD$ with $P, E, N,$ and M midpoints of its sides.

To determine the kind of a parallelogram formed by joining these points.

Since the diagonals of a general paral-

leogram are unequal, the adjacent sides of the parallelogram will be unequal.



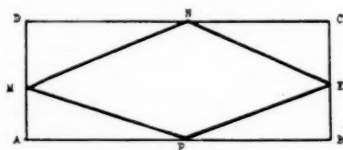
And since the diagonals intersect at oblique angles, the angles of the parallelogram will be unequal.

Therefore, the parallelogram $PENM$ will be a general parallelogram.

CASE IV

Given: Rectangle $ABCD$ with P , E , N , and M midpoints of its sides.

To determine the kind of a parallelogram formed by joining these points.



Since the diagonals of a rectangle are equal, the adjacent sides of the parallelogram will be equal.

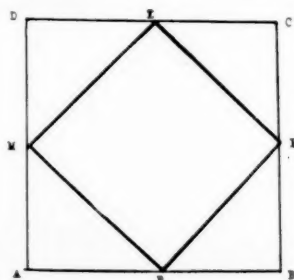
And since the diagonals intersect at oblique angles, the angles of the parallelogram will be unequal.

Then the parallelogram $PENM$ will be a rhombus.

CASE V

Given: The square $ABCD$ with P , E , N , and M midpoints of its sides.

To determine the kind of a parallelogram formed by joining these points.



Since the diagonals of a square are equal, the adjacent sides of the parallelogram will be equal.

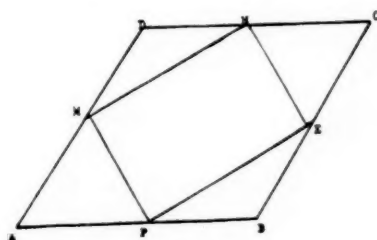
And since the diagonals of a square intersect at right angles, the angles of the parallelogram will be right angles.

Therefore, the parallelogram $PENM$ will be a square.

CASE VI

Given: The rhombus $ABCD$ with P , E , N , and M midpoints of its sides.

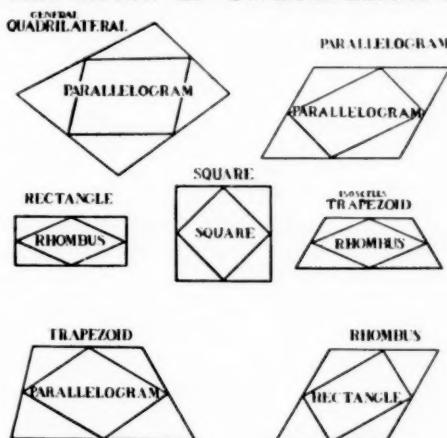
To determine the kind of a parallelogram formed by joining these points.



Since the diagonals of a rhombus are unequal, the adjacent sides of the parallelogram will be unequal.

And since the diagonals intersect at right angles, the angles of the parallelogram will be right angles.

FIGURES FORMED BY JOINING MIDPOINTS OF QUADRILATERALS



Therefore the parallelogram *PENM* will be a rectangle.

These short concise proofs involving functional thinking or dependence are better than the longer method of congruency.

Additional units may be developed by bisecting the angles, drawing the alti-

tudes, and the perpendicular bisectors of the sides of all quadrilaterals and then determining the nature of the figures formed.

This unit may be summarized by means of the poster at the bottom of the right-hand column on page 71.

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The Lessons Non-Euclidean Geometry Can Teach

By KENNETH B. HENDERSON

Rocky River High School

Rocky River, Ohio

So you want to teach mathematics! Why you selected college geometry is more than I know because college geometry is about as useless as high school geometry. Seriously now, did the fact that an exterior angle of a triangle equals the sum of the two non-adjacent interior angles ever make any real difference in your behavior? And were you thrilled to find out that a line is tangent to a circle if it is perpendicular to the radius drawn to the point of contact? Let me warn you in advance that when you find out a line is perpendicular to a plane if it is perpendicular to all lines in the plane passing through its foot, you will be apathetic. Furthermore, when you prove that the volume of a rectangular solid is really the product of the length, width, and height you will probably be disgusted, for you knew that ever since you were in the seventh grade. The fact that you proved it is really silly, isn't it? You have gotten along pretty well so far just taking somebody's, a teacher's or author's, word for it. And after all, what difference does it make whether you assume this is so in the first place or whether you assume something else, and from this prove the theorem for finding this volume. None of you would try to make me believe that you can calculate the area of the floor of this room better than Fred Fable whose schedule conflicts prevented him from being one of your classmates in geometry in high school.

I won't ask you why you elected to take college geometry because I already know the stereotyped answers: "I need eighteen hours of mathematics for a teaching major," "I liked plane geometry, and got good marks in it so I might as well continue it in college," "Since I'll have to teach geometry in some high school, I might as well learn more about it." You

know when I took geometry we thought we had the answer to why we took the subject—or thought we did. Geometry trained the mind to reason. Didn't you always "prove" every statement by backing it up with reasons? *Ergo*, you learned to reason. This was a real justification for geometry to me until I read a book by an ex-confidence man who stated that teachers were always prominent members of any "sucker list" that was compiled. This started me to observing, and much to my consternation I couldn't find that geometry students were any more rational than any others. Even geometry teachers, as close as they are to the Grail, are no shining examples of clear thinkers. Just recently in a questionnaire study at Ohio State University, science and mathematics teachers were found to be most sure of their decisions and also most inconsistent in their reasoning. All of this seems to indicate that a student passing through geometry learns to think in spite of the subject rather than because of it. That this is undoubtedly so, Thorndike, Watson, and other famous psychologists have proved more scientifically than I. You are probably at the point of asking why we still teach the darn stuff. Well, partly because of inertia, and partly because some of us believe that geometry, if presented correctly, can give students a real insight into the nature of proof. Do you think you would be more intelligent in your behavior if you could size up an argument, formal or otherwise, and know whether it is sound and consistent or illogical and based on faulty premises? Assuming you do, we are going to try to learn to do that very thing in this course. Perhaps then you can pass it on to your pupils.

Let's start the ball rolling by taking a look at definitions: what they are, the part

they play, and what would happen if we changed them a bit. Do you remember you said in high school geometry that a straight line is the shortest distance between two points? There wasn't any argument about that, was there? And if there were, obviously it would all depend upon what we mean by a "straight line." This may seem like quibbling to you, for you think you know what a straight line is. Yes, I see a lot of you are looking at the edges of the floor boards, and some at the junction of the walls and the ceiling. And if I dropped this piece of chalk, it would fall straight down, wouldn't it? For our purposes it would, but while it is falling the earth is rotating on its axis and with it this building including the chalk, so that to someone standing on one toe on the North Pole the chalk would appear to fall along AC rather than along AB as it seems to us. To



FIG. 1

the astronomer on Arcturus who sees our solar system moving, the earth revolving around the sun, and rotating on its axis all while the chalk is falling, the path of the chalk would be some complex curve which I won't attempt to draw. In all events the Arcturian astronomer certainly wouldn't call it straight.

Consider now the shortest distance between two points on the surface of a sphere such as our earth. Since it is practically impossible to draw lines through the solid earth, the shortest distance between these two points, so far as any practical value is concerned, must lie on the surface of the sphere. For example, we say the distance between New York and Los Angeles is 3111 miles, and we know that this distance is measured on the surface of our

earth, which is almost a sphere. Accordingly, a straight line, which we defined as the shortest distance between two points, on a sphere is a curve. Actually it is the arc of a great circle passing through the two points. Thus the equator and the meridians are great circles, and paradoxically, straight lines.

It now becomes apparent, doesn't it that the nature of the shortest distance and consequently a straight line depends upon the kind of surface with which we are dealing. We could give the shortest distance between two points another name: geodesic, and remember that its shape depends upon the surface on which it is drawn. On a Euclidean plane a geodesic is one kind of line, on a sphere it is the arc of a great circle, on a pseudo-sphere, which looks like two long trumpets soldered together at the big ends, it is another kind of a curve. To sum it all up, what is a straight line to one man is not a straight line to another, depending where he is or what point of view he holds.

In order then to believe a person when he says, "A triangle is a three-sided figure whose sides are all straight lines and the sum of whose angles is 180 degrees," we must know what he means by straight lines. If they are Euclid's straight lines, the statement is true. If they are the straight lines of a sphere or a pseudo-sphere, his statement is false as we shall see later. In the same way when candidate X speaks extensively of "adequate state support for our great system of free public education which is the bulwark of American democracy" we needs must know what "adequate support" means. Before you believe in "avoiding foreign entanglements" you had better decide whether buying abroad, owning foreign securities, entering the League of Nations, and traveling abroad are "foreign entanglements."

Don't let words conceal the absence of thought. Define precisely and religiously; the clarification of thinking that is a natural concomitant is worth the pains.

Now let's change an old familiar defini-

tion a bit and see what happens. Euclid said that parallel lines were lines in a plane which did not meet no matter how far they were extended or something like that. Parallel lines are very real to us. The only time we begin to doubt that there are lines which do not meet is when we look down the railroad tracks and see the rails apparently converging. There was a man by the name of Riemann who had the audacity to take care of the whole business of parallels by simply saying, "In my geometry there can be no such lines, for all lines are of finite length and return to their origin." In spite of the fact that parallel lines had existed since the time of Euclid, some 2150 years, and almost everyone was sold on this idea, no one was able to prove Herr Riemann wrong. This was because a person has a logical right to define his terms anyway he pleases so long as he is consistent in the use of these definitions.

But isn't the assumption that through a point outside a given line no line can be drawn parallel to the given line absurd? This is what Riemann's rejection of the definition of parallels amounts to saying. It is absurd if you think of Riemann playing the game on Euclid's field, the Euclidean plane. It is most sensible and logical if we allow Riemann to choose the field. We have already found that straight lines or geodesics on a sphere are arcs of great circles. If two of these geodesics are extended sufficiently, they will always intersect, not once but twice.

Let me anticipate what you are going to say about Riemann's failure to recognize that parallels of latitude are parallel lines on a sphere even though the lines are curves. Sorry, but you are wrong. You remember we began by defining straight lines as geodesics or the shortest distance lines. Parallels of latitude are therefore not straight lines or geodesics because they are small circles, that is, circles formed by Euclidean planes passing through the sphere in such a manner that they do not pass through the center of the sphere. Consequently small circles on a sphere are

somewhat analogous to curves on a Euclidean plane and statements concerning "lines" must omit these from consideration. So the conception of the non-existence of parallel lines is perfectly thinkable provided we visualize the proper field or plane and the proper kind of "straight lines" or geodesics.

By drawing geodesics on the sphere we can form triangles, quadrilaterals, and polygons, and study the geometry of the plane we call the surface of a sphere. Suppose we draw two lines perpendicular to a given line at different points. See Fig. 2.

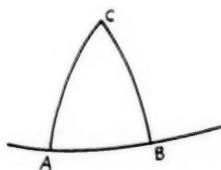


FIG. 2

Since all lines on a sphere eventually intersect, the perpendiculars AC and BC will meet to form a triangle ABC . But angles A and B are 90 degrees each. This makes the sum of two angles of the triangle 180 degrees without considering the other angle C of the triangle. Hence the startling theorem: The sum of the angles of a triangle in the Riemannian plane is always greater than 180 degrees. Imagine—a triangle with three right angles.

Suppose we draw an isosceles birectangular quadrilateral $ABCD$ on the Riemannian plane as in Fig. 3. You will

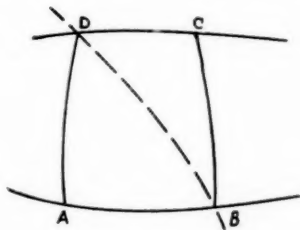


FIG. 3

just have to take my word that it can be done. The right angles are at A and B and AD is equal to BC . If a straight line is

drawn through points B and D , two triangles are formed. Now the sum of the angles of the two triangles and hence the sum of the angles of the quadrilateral is in excess of 360 degrees. Subtracting 180 degrees, the sum of the base angles A and B , we have the sum of angles D and C greater than 180 degrees. Since these two angles are equal, each must be greater than 90 degrees or each must be an obtuse angle. According to Euclid's rules, each would be equal to 90 degrees or a right angle. Also by Euclid's rules, DC would equal AB , but according to Riemann's DC is evidently less than AB .

None of these theorems which have

jected the concept of parallels and assumed it impossible to draw a line through a given point parallel to a given line, it might be possible to define parallel lines in such a way that through the point two lines could be drawn parallel to the given line. About 1828 a Russian professor, Nicholas Lobachewski, chose to change the accepted definition of parallel lines. He took (Fig. 4) P as any point not on the straight line AB . Through P there will pass some lines such as LPJ which will intersect AB if extended far enough, and some lines such as GPH which will not meet AB no matter how far they be extended. In such case there will conceivably be one line, EPI

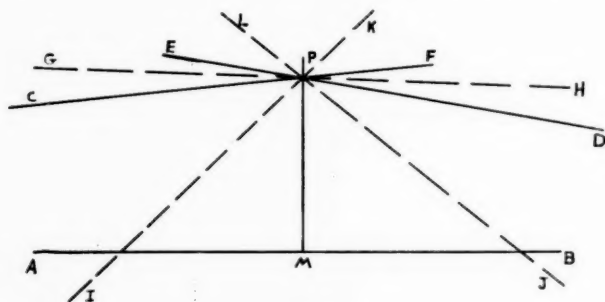


FIG. 4

been stated could bear the close scrutiny of the demands of rigor without blushing. I am only attempting to give you a few reasons which substantiate them. The point to be remembered is that they could be proven to any one's satisfaction provided a person wanted to study this subject more completely. Other theorems which can be proved and are perfectly logical in Riemannian geometry are: The sides opposite the equal angles of a triangle are equal, triangles having the same angle sum are equal in area, and an exterior angle of a triangle is not always greater than either non-adjacent interior angle. You can see that some conclusions are the same on both planes, and yet some are different. Such is the result of a change in a definition.

Maybe some of you fast thinkers are already speculating that, since one man re-

which will be the border line between these two sets of lines. Extending this reasoning to the two sets of lines, intersecting and non-intersecting, which extend in the other direction of which KPI and HPC are respectively examples, FPC may be represented as the border line. These two lines, one in one direction and another in the opposite direction, Lobachewski chose to call parallel to the given line AB . In his own words he said (or so his translator says he said), "All straight lines which in a plane go from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *non-cutting*. The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*."¹ So we

¹ Nicholas Lobachewski. *Geometrical Researches on the Theory of Parallels*, Translated from the original by George Bruce Halstead.

have two lines through a point parallel to a given line.

As you probably guess Lobachewski must have a plane for his geometry which differs from those of Euclid and Riemann. After Lobachewski develops his geometry to some extent, it becomes evident that his plane is one of constant negative curvature, the orisphere or pseudo-sphere about which we have already heard. If in Fig. 4 PM is taken perpendicular to AB , the angle DPM , which is called the angle of parallelism, is taken to be less than 90 degrees. It then follows that in case the angle of parallelism is 90 degrees, the two lines will never meet. In other words on a pseudo-sphere two lines which have a common perpendicular neither intersect nor are parallel. Euclid finds such lines parallel; Riemann finds they intersect.

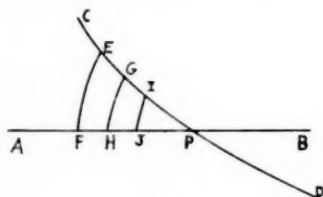


FIG. 5

It can be proved that two lines which are not parallel eventually diverge. Let us consider first the case of two lines which eventually intersect. EF in Fig. 5 is either equal to, greater than, or less than GH . Certainly EF cannot equal GH , for if it did, midway between F and H the shortest distance from AB to CD , the common perpendicular, could be drawn. This is not possible because two lines which have a common perpendicular do not intersect as we have already shown above, and these two lines do intersect by hypothesis. EF cannot be less than GH because if it were, we could locate some line such as IJ , which is equal to EF , by going very close to P , the point of intersection. But this again would give a common perpendicular. Why this cannot be has just been ex-

plained. It must be then that on a pseudo-sphere two intersecting lines diverge more and more just as they do on Euclid's plane. Two lines which are not parallel and do not intersect also diverge once they have passed their common perpendicular. If AB and CD represent two such lines, (Fig. 6) it is possible to draw through some

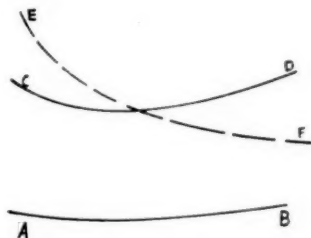


FIG. 6

point on CD a line EF parallel to AB . Now we have just proven that CD and EF , two intersecting lines, diverge. Since EF and AB approach each other more and more, AB and CD must ever diverge.

Categorically speaking then, the lines on a pseudo-sphere are of infinite length and eventually either diverge or approach each other asymptotically. This may be compared to the behavior of lines on Riemann's sphere and Euclid's plane. A change in a definition produces some unexpected results.

Let's take a look at an isosceles birectangular quadrilateral drawn on a pseudo-sphere, and compare our findings with those on a sphere and Euclidean plane. From the assumption that the angle of parallelism is less than 90 degrees, it follows that the farther parallel lines are extended in the direction of parallelism, the more they approach each other, so that in the birectangular quadrilateral $ABCD$ (Fig. 7) in which AB is parallel to CD and AD and BC are perpendicular to AB , both angles at the vertex; i.e., D and C are acute. Furthermore, CD is greater than AB . How do these findings compare with those on the other two planes? Let's pursue this a bit further. Let ABC be a triangle on the Lobachewskian plane with

GF joining the midpoints of the sides AC and BC . AE , CH , and BD are all perpendiculars dropped from the vertices of the

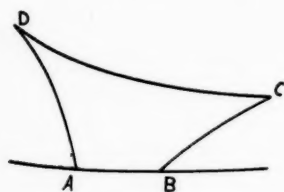


FIG. 7

triangle to the line GF prolonged. By proving triangles AEG and CGH , and CHF and BDF congruent (and by the way, the Euclidean theorem used to do this happens to be equally true on the pseudo-

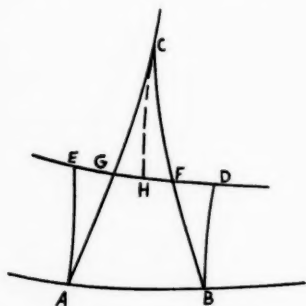


FIG. 8

sphere) we get $AE = CH = BD$. Therefore, $ABDE$ is an isosceles birectangular quadrilateral whose summit angles are EAB and FDB . From the congruent triangles we get angle $EAG = \text{angle } GCH$ and angle $DBF = \text{angle } FCH$. Therefore, angle ACB , which is equal to the sum of angles GCH and HCF , is equal to the sum of angles EAG and DBF . Adding angles GAB and FBA to both sides of the equation, angle $ACB = \text{angle } EAG + \text{angle } DBF$, we find that the sum of the angles of the triangle ABC equals the sum of the two summit angles of the birectangular quadrilateral which sum in turn we have shown to be less than 180 degrees. Hence we have the theorem: The sum of the angles of a triangle on a pseudo-sphere is less than 180 degrees.

If we wanted to pursue our proofs further, we could prove such theorems as: the sides opposite the equal angles of a triangle are equal, triangles which have the same angle sum are equal in area, and an exterior angle of a triangle is less than either non-adjacent interior angle. Compare these conclusions with analogous conclusions in the other two systems.

It's time we bring all this to a head because by now you either are bored to tears and can't see the woods for the trees, or you are excited about these different geometries, and may plunge in so deeply that you, too, will miss the main point. It's not the conclusions that these various men reached; for our purposes they don't amount to a rap. The thing to keep your finger on is that the conclusions are different. Since the same laws of logic are used in evolving the conclusions in each case, and since the only difference among the three is the differing ways of handling the concept of parallels, the difference in the conclusions must be due to the difference in the original definitions. Euclid defined parallel lines as lines which never meet no matter how far they are extended. From this it follows logically that the sum of the interior angles of a triangle is 180 degrees. Lobachewski defines a line parallel to a given line as the boundary line between the set of lines which intersect the given line and the set which do not meet the given line. As a result he finds that the sum of the interior angles of a triangle is less than 180 degrees. The sphere may be taken as the field on which Riemannian or elliptic geometry holds fairly well; a pseudo-sphere does about as well for Lobachewskian or hyperbolic geometry, and the conventional plane for Euclidean or parabolic geometry as it sometimes is called. "Straight lines" become, speaking in terms of the most common geometry, straight lines on a plane, circles on a sphere, and tratrixes on a pseudo-sphere. It all depends upon the definitions you make in the first place. Definitions would seem to be very important, for if you ac-

cept a person's definitions you must, provided you are intellectually moral, accept the conclusions which logically grow out of them.

Now, I wonder whether when you become teachers you can't present at least one unit in the geometry course in high school dealing with the nature and importance of definitions. Teach your pupils in the first place to define carefully the terms which they use in life. Sterile thoughts can only result from empty concepts. Teach them in the second place to demand precise definitions from others; to be alert for undefined terms and "weasel words" which find their way into generalizations; to realize that points of view which differ as markedly as those of democracy and totalitarianism depend in part upon the different denotations of such terms as so-

cial rights, equality of opportunity, duty, intelligence, cooperation, and way of life.

You will be directors of a block of student time which tradition has labeled "geometry," but over which tradition has no supreme control. Do something more than merely pay lip-service to "teaching pupils to think" as an aim in education. Geometry as a course whose integrating theme is the nature of proof one unit of which deals with definitions would be efficacious in underwriting such an aim. This theme is universal; the subject matter and experiences for your students should sample all fields. Such a procedure should rejuvenate a decadent subject the teaching of which at present would be a travesty if the waste of a period of student's time each day for nine months were not such a serious matter.

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The Trisector of Amadori

By MARIAN E. DANIELLS
Iowa State College, Ames, Iowa

It is well known that, although the Greeks searched unsuccessfully for a method of trisecting an angle by means of an unmarked straight edge and a compass only, they did devise methods of trisecting by other means. One of the oldest of these methods, which is attributed to Archimedes was known as a "*veusis*" or verging" and consisted essentially of making a line approach a specified length or position.

Q. Amadori devised a simple instrument which is based upon a "*veusis*," which is easily made, and which gives quite satisfactory results if the angle to be trisected is not too small. It is based upon the following geometrical consideration.

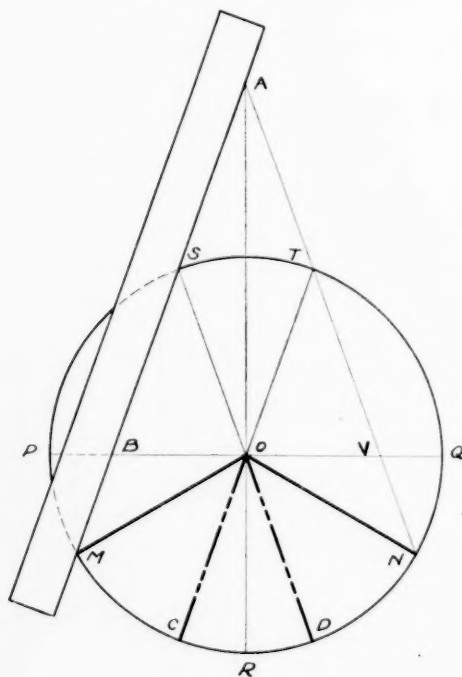


FIG. 1

If an angle MON (Figure 1) is the angle to be trisected, let us bisect it by the line OR , with any length as a radius and the

vertex O as a center draw circle, and construct a diameter PQ perpendicular to the bisector. Mark off on a straight edge a distance AB equal to PQ and place the ruler on the plane of the angle so that the point B falls on the diameter PQ and the

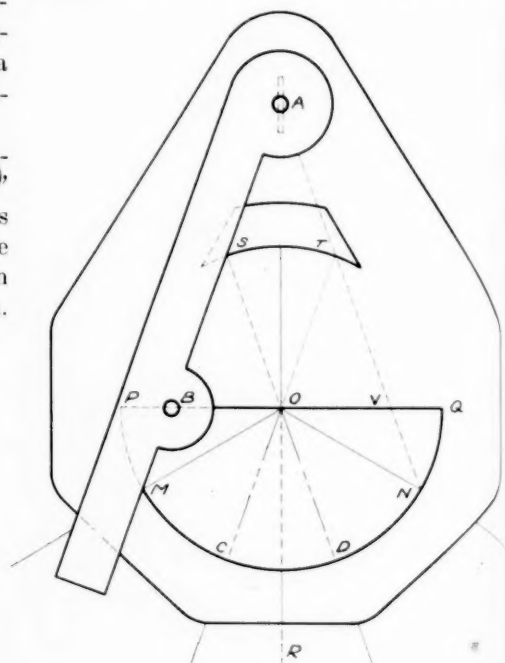


FIG. 2

point A falls on the bisector OR . Then shift the ruler until its edge passes through the point M , keeping B on the line PQ and A on the line OR . Mark the point of intersection of the ruler and the circle S . By following a similar procedure with the point N locate the point T . Draw the lines TO and SO and produce them to intersect the circumference at C and D . These diameters, TOC and SOD trisect the angle.

Amadori's instrument may be made of heavy cardboard, brass, wood, or celluloid. The model shown in Figure 2 is of sheet celluloid. It consists of two pieces,

one of which is a nine-sided polygon symmetrical to one axis. The cut-outs in this piece are a semicircle, a "mixed-line" isosceles trapezoid, and a rectangle. The arc of the trapezoid cut-out subtends a central angle of 60 degrees and the axis of the nine-sided polygon is etched on the celluloid. The other piece in the form of a ruler is attached to the first piece by bolts. The bolt *A* can slide in the rectangular cut-out and the bolt *B* fastens the ruler to the edge *PQ* of the semi-circular cut-out. The distance between the axes of these bolts equals the diameter *PQ* and the right edge of the ruler goes through the axes of the bolts, both of which can be screwed tightly in any position.

In order to trisect a given angle *MON* one bisects it and then places Amadori's trisector on the plane of the angle so that the center *O* coincides with the vertex of

the angle and the axis coincides with the bisecting line. The bolt *B* is then shifted along the edge *PQ* until the right hand edge of the ruler passes through *M*, the intersection of one side of the angle with the semicircle. Now set the bolts tight, mark the intersection, *S*, of the line *AM* with the circle, and draw line *SOD*. By a similar process determine the line *TOC*. These lines *SOD* and *TOC* trisect the given angle.

For $SO = \frac{1}{2}AB$

angle *AOB* = 90 degrees,

$\therefore AS = SO = SB = OT = TV = TA$,

and *SOTA* is a rhombus.

$SO = AT$ and $TO = SA$,

\therefore angle *BSO* = angle *OTV* = angle *COD*,

angle *COD* = angle *DON* = angle *MOC*.

TEN YEARS

Mathematics Classroom Instruments and Supplies

Lafayette Instruments Inc., *The Mathematics House*, has been manufacturing and selling inexpensive instruments and supplies for High School Mathematics Departments since 1930. A year ago Yoder Instruments, an associate company, was established at East Palestine,

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◆ THE ART OF TEACHING ◆

Analytic Geometry and Trigonometry in Second Year Algebra

By WINSTON M. GOTTSCHALK

St. Mark's School, Southborough, Massachusetts

IN PREPARATORY schools the modern trend is to begin geometry in the latter half of the second year, finishing algebra in the third year. In smaller schools such as ours a student sometimes takes geometry and second year algebra simultaneously. Such arrangements give a welcome opportunity to bring out the union of the two subjects in analytic geometry. My experience has been that such procedure is easily carried out and that the interest of the students is aroused. Then, too, certain phases of both subjects are given meaning and the student is prepared for the geometric illustrations of analysis, so helpful in studying calculus.

From the moment graphs are first encountered I stress the one-to-one correspondence of the points on a plane and number pairs. From this concept to finding and representing algebraically the distance between any two points is but a step. Coupled with the idea of locus one is led naturally to the equation of a circle with the origin as center. This I next amplify by displacing the center along first one axis and then the other, ending with the general equation for the circle.

With the straight line the easiest beginning is the recognition of the condition for parallelism, which leads to the slope form of the equation. Next I demonstrate the intercept form of the equation and the two point formula. I have given the problem of obtaining tangents to a circle parallel to a given line as an original exercise with good results. This problem has the added value of bringing in the solution

of a system of equations, one linear and the other quadratic. Solving this problem makes sensible the sets of problem on the quadratic discriminant such as " $x^2+kx+k^2-1=0$, determine k so that the roots shall be equal." I feel strongly that when such problems receive a meaning and a purpose they rise above drill and one is teaching mathematics, not drilling as an exercise.

With the conic section I bring out at an early stage the ideas of symmetry to make curve drawing less mechanical. I also try to have a student analyze the equation to see what becomes of x as y varies and to get a sense of the general trend of the curve before plotting it. This helps him to choose points at advantageous positions. Before long the majority of the class can tell at sight which one of the curves will be obtained.

More or less of this work can be done in accordance with the aptitude of the class. The ordinary course in algebra allows ample time for the introduction of such subject matter if it is done systematically as the class reaches the stages natural to such introduction. We teachers should try to remember that there are always the two approaches, analytic and geometric, and often a student is poor because his mind takes the other way from the one we teach. I understand that the great Sophus Lie was quite inept at the customary forms of mathematics.

I feel that it is almost negligent not to introduce some trigonometry in addition to the usual bit. Fusion courses and text-

books have been successful, but my procedure has not been so extensive. However, when a student has learned the functions of an angle and can solve right triangles, what is more natural than the derivation of the Pythagorean formulas $\sin^2 x + \cos^2 x = 1$, etc.? These can usually be done for homework, and I have even given them on hour quizzes. To students taking geometry the derivation of the law of cosines is simply another way of expressing one of the formulas developed in geometry. The derivation of the law of sines is easy and is an excellent example of the mathematician's method of building up formulas and breaking into new ground. With this much as a foretaste many students will elect to continue their mathematics into fourth year and be eager to study trigonometry.

It seems to me that we who teach mathematics should seek to initiate young minds into the methods of mathematical thinking and to give them as early as pos-

sible the pleasure of working through an absorbing problem more or less independently to a result which is, for them, a discovery. I can scarcely blame a student for disliking secondary-school mathematics when it is composed largely of drill problems which carry no meaning and are to him a waste of time. It is true that in the interests of technique a goodly amount of such work is necessary, but when the opportunity presents itself we should embark the students on little researches of their own and let them have some of the pleasure of doing mathematics.

In all this work I stress the principle of going back to several things we know and of combining them to attack problems that seem to lie beyond our power. This implants a sense of independence and courageous attack. This is the method by which mathematics progresses and in grasping this idea the students gain an insight into all of mathematics.

Plays

BACK numbers of THE MATHEMATICS TEACHER containing the following plays may be had from the office of THE MATHEMATICS TEACHER, 525 West 120th Street, New York, N.Y.

A Problem Play. By Dena Cohen. XXIX, Feb. 1936.

The Eternal Triangle. By Gerald Raftery. XXVI, Feb. 1933.

Out of the Past. By Florence B. Miller. XXX, Dec. 1937.

Alice in Dozenland. By W. E. Pitcher. XXVII, Dec. 1934.

Everyman's Visit to the Land of the Mathematicians. By Edith B. Paterson. XXXI, Jan. 1938.

When Wishes Come True. By Hannah A. Parkyn. XXXII, Jan. 1939. Price: 25¢ each.

Program, National Council of Teachers of Mathematics

Hotel Chase, St. Louis, February 22, 8:30 A.M. to 23, 9:30 P.M., 1940

Theme: Mathematics for the Other-than-College-Preparatory Student

I. General Meetings. Regency Room

1. February 22, 4:30 to 5:30 p.m.
Get-acquainted and see-exhibits hour. Reception committee in charge.
2. February 22, 8:00 p.m.
Presiding: H. C. Christofferson, Miami University, Oxford, Ohio.
Address of Welcome: John Rush Powell, Assistant Superintendent of St. Louis Schools.
"Modern Youth Challenge the Curriculum" J. Paul Leonard, Stanford University.
"Mathematics in Education as Viewed by a School Administrator," John L. Bracken, Superintendent of Schools, Clayton, Mo.
"Mathematics in and for the Modern Curriculum," W. D. Reeve, Teachers College, Columbia University.
3. February 23, 8:30 to 10:00 a.m.
Annual Business Meeting.
4. February 23, 12:15 to 3:00 p.m.
Luncheon for Representatives, Delegates, and Directors. A. E. Katra in charge. \$1.10 per person.
5. February 23, 1:00 to 3:00 p.m.
Topic: *Visual Aids*.
Presiding: E. R. Breslich, University of Chicago.
"The Use of Films in Motivating the Study of Mathematics," The Chevrolet Motor Company.
"The Use of Slides and Films in Correlating Mathematics with Other Fields," Ida Fogelson, Bowen High School, Chicago.
"The Use of Films in Supplementing the Teaching of Mathematics," The Chicago Model Club and the schools of St. Louis and vicinity have on display Thursday and Friday, an exhibit of varied visual aids.
6. February 23, 6:00 to 8:00 p.m.
Discussion Banquet. \$2.25 per person. Informal.
Program, 8:00 to 9:00 p.m.
Music—Glee Club, John Burroughs School, Ralph Weinrich, Director.
Address: Functional Figures, C. A. Hutchinson, University of Colorado, Boulder.

II. Elementary School Programs. Roof Garden

1. February 22, 10:15 to 11:45 a.m.
Residing: Jesse Osborn, Harris Teachers College, St. Louis.
"Analytic Analyses in Arithmetic Processes," Irene Sauble, Supervisor of Exact Sciences, Detroit.
Discussion Leader, Helen Turley, Wieman School, St. Louis.
2. February 22, 2:00 to 4:00 p.m.
Presiding: Foster Grossnickle, State Normal School, Jersey City, N.J.
"Number Readiness," Josephine MacLatchy, Bureau of Educational Research, Ohio State University, Columbus.
Discussion Leader: Marguerite Versen, Primary Supervisor of St. Louis Schools,
3. February 23, 10:00 to 11:45 a.m.
Presiding: C. L. Thiele, Director Exact Sciences, Detroit, Mich.
"The Nature and Purpose of Consumer Mathematics," H. R. Risinger, School of Education, Rutgers University.
Discussion Leader: Jesse Osborn.
4. February 23, 3:00 to 5:00 p.m.
Presiding: C. L. Thiele.

"Psychology and Its Arithmetic Applications," T. R. McConnell, Professor Education and Chairman of Commission of Educational Research, University of Minnesota, Minneapolis.

Discussion Leaders: Stephen Gribble, Washington University, and George R. Johnston, Director of Tests and Measurements in the St. Louis Schools.

III. Secondary School Programs. Regency Room and Chase Room

1. February 22, 8:30 to 10:00 a.m.

Topic: *Mathematics for the High School Pupil Not Going to College.*

Presiding: Mary A. Potter, Racine, Wis.

"Who Is the Non-College Preparatory Pupil?" Virgil Mallory, State Teachers College, Montclair, N.J.

"Mathematics and Life Advisement," Theo Donnelly, Milwaukee.

"Report of the Joint Commission," Ernest R. Breslich, University of Chicago, Chicago.

2. February 22, 10:15 to 11:45 a.m.

Topic: *Consumer Education and Mathematics in the Junior and Senior High Schools.*

Presiding: J. Paul Leonard, School of Education, Stanford University.

"Mathematics in a Democracy," Edith Woolsey, Sanford Junior High School, Minneapolis, Minn.

"Consumer Education in the Junior High School," William H. Garrett, Webster Groves High School, Webster Groves, Mo.

3. February 22, 2:00 to 4:00 p.m.

Topic: *Non-College Preparatory Mathematics in the Senior High School.*

Presiding: Maurice L. Hartung, University of Chicago.

"The Mathematics Everyone Needs," Kate Bell, Lewis and Clark High School, Spokane, Wash.

"Mathematics in the Training of Industrial Workers," Clarence Leonard, Southeastern High School, Detroit.

"Five Years of Experiments in Evanston Township High School," Clara Murphy, Evanston Township High School, Evanston, Ill.

Discussion.

4. February 22, 2:00 to 4:00 p.m.

Topic: *Non-College Preparatory Mathematics in the Junior High School.*

Presiding: William Betz, Rochester, N.Y.

"The Changing Junior High School and Non-College Mathematics," H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa.

"Cultural Material," Ruth Lane, University of Iowa, Iowa City.

"Do We Try to Teach Too Many or Too Difficult Topics, or Both?" Hildegarde Beck, McMichaels Intermediate School, Detroit.

"Mathematics for the Capable Pupil Not Going to College," Florence Brooks Miller, Shaker Heights, Ohio.

Discussion.

5. February 23, 10:00 to 11:45 a.m.

Topic: *What Mathematics Can Contribute to the Education of Modern Youth.*

Presiding: M. F. Rosskopf, John Burroughs School, Clayton, Mo.

"Mathematics for the College-Preparatory Student," Gordon Browne, St. Louis Country Day School, St. Louis.

"Mathematics in a Progressive School," Lewis Taylor, North Shore Country Day School, Winnetka, Ill.

"Mathematics in University High Schools," Ruth Lane, University High School, Iowa City.

6. February 23, 10:00 to 11:45 a.m.

Topic: *What Mathematics Can Contribute to the Education of the Investor and the Contractor.*

Presiding: C. A. Smith, Southwestern High School, St. Louis.

"Applications of Trigonometry in Surveying," Charles A. Hutchinson, University of Colorado, Boulder, Colo.

"Education of the Future Investor," From the viewpoint of the broker,

- H. Lloyd Kelley, of the firm of Paul Brown and Co., Brokers, St. Louis. From the viewpoint of the banker, Byron Moser, President, Mutual Bank and Trust Company, St. Louis. From the viewpoint of the insurance man, Fred T. Rench, President, St. Louis Life Underwriters Association and General Agent of the National Life Insurance Company of Vermont.
7. February 23, 3:15 to 4:30 p.m.
 Demonstration Lesson.
 Presiding: Jesse Osborn, St. Louis.
 "Social Mathematics in the High School," Clarabel Parks, Beaumont High School, St. Louis.
8. February 23, 3:00 to 5:00 p.m.
 Special conference on social mathematics in grades eleven and twelve: an exchange of experiences by those who have taught or designed such a course.
 Presiding: Raleigh Schorling, University of Michigan, Ann Arbor, Mich.
 This is a limited group meeting. Secure admission by writing or seeing Dr. Schorling or Dr. Christofferson.

IV. Teacher Education Programs. Colonial Room

1. February 22, 10:15 to 11:45 a.m.
 Topic: *The Training of Teachers of Algebra*.
 Presiding: G. H. Jamison, State Teachers College, Kirksville, Mo.
 "The Academic Preparation of Teachers of Algebra," J. O. Hassler, Oklahoma University, Norman, Okla.
 "The Professional Preparation of Teachers of Algebra," A. R. Congdon, University of Nebraska, Lincoln.
2. February 22, 2:00 to 4:00 p.m.
 Topic: *The Training of Teachers of Geometry*.
 Presiding: L. H. Whitcraft, Ball State Teachers College, Muncie, Ind.
 "What General Knowledge and Skills Should the Teacher Training Program Provide?" L. D. Haertter, John Burroughs School, St. Louis.
 "What Mathematical Knowledge and Abilities Should the Teacher Training Program Provide in Fields Other Than Geometry?" Gertrude Hendrix, Eastern Illinois State Teachers College, Charleston.
 "What Specialized Knowledge Should the Teacher Training Program Provide in the Field of Geometry?" P. D. Edwards, Ball State Teachers College, Muncie.
 "What Professional Knowledge and Experiences Should the Teacher Training Program Provide?" C. N. Mills, Illinois State Normal University, Normal.
3. February 23, 10:00 to 11:45 p.m.
 Topic: *The Training of Teachers of High School Arithmetic and General Mathematics*.
 Presiding: F. L. Wren, Peabody College for Teachers, Nashville, Tenn.
 "Training Teachers of High School Arithmetic," Judson W. Faust, Central State Teachers College, Mt. Pleasant, Mich.
 "Training Teachers of General Mathematics," Charles H. Butler, Western State Teachers College, Kalamazoo, Mich.
 "Training Teachers of Mathematics to Comprehend and Appreciate the Interdependence of Different Mathematical Fields," Edwin A. Beito, University of Wichita, Wichita, Kan.

Hosts and Hostesses for the Banquet

It is the plan of the committee in charge of the banquet at 6 o'clock, Friday evening, February 23, to have a host at each table who will attempt to direct the discussion at the table around the topic suggested. In making your reservation, choose the host and the topic which you would like to discuss during the luncheon. Send your reservation to Miss Annabel Remnitz, Roosevelt High School, St. Louis, Mo. State the table of your first, second, and third choice.

Table

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Table	Host	Topic
1.	E. A. Beito	Mathematics of Air Navigation
2.	William Betz	Contrasting the Point of View of the Joint Commission and the Progressive Education Association
3.	E. R. Breslich	Trends in the Teaching of Mathematics
4.	A. R. Congdon	Outcomes of High School Mathematics Valuable for Anyone
5.	Theo E. Donnelly	How Milwaukee Provides for Pupils Not Going to College
6.	Judson Foust	What is Mathematics?
7.	Maurice Hartung	Method of Creating Interest in Mathematics
8.	J. O. Hassler	Teaching Geometry So As to Encourage Independent Thinking
9.	G. H. Jamison	Mathematics and Creative Education
10.	Ruth Lane	An Exchange of Devices
11.	Virgil Mallory	How Does Plane Geometry Teach Logical Thinking?
12.	A. Brown Miller	How Attain Better Guidance in the Selection of Mathematics Courses?
13.	Mary A. Potter	How Our Schools Serve the Mathematical Needs of Our Town
14.	W. D. Reeve	Mathematics
15.	Vera Sanford	"Let's Just Talk"
16.	Irene Sauble	Distinctive Features of a Course of Study in Arithmetic Based on Generalizations
17.	W. S. Schlauch	Calibrating the Difficulty of Algebra or Geometry Problems
18.	R. R. Smith	Eleventh Year Mathematics
19.	C. L. Thiele	Number Meaning Through the Use of Concrete Material
20.	L. H. Whitcraft	Securing Interest in Ninth Grade Algebra.
21.	Edith Woolsey	Socialized Mathematics
22.	F. L. Wren	What Mathematical Training for Teachers of Mathematics?
23.	C. R. Atherton	A Program on Consumer Mathematics for the Twelfth Grade.

A FEW words now about the final problem, trisecting an angle. This has received a good deal of attention in the newspapers during the past few years on account of the fact that several teachers of mathematics in high schools and colleges in this country have claimed that they have solved the problem completely.

It turns out, however, that all these published solutions are absolutely incorrect. The error committed is usually one of four kinds: sometimes the solution is merely approximate instead of exact; sometimes instruments other than the ruler and the compass are used, consciously or unconsciously; sometimes there is a logical fallacy in the pretended proof; sometimes only special arbitrary or general angles are considered. New trisectors will always appear, but newspapers should not give them any further attention.—EDWARD KASNER, Professor of Mathematics, Columbia University. From "Science Service Radio Talks," *The Scientific Monthly*, July, 1933, Vol. XXXVII, pages 67-71. Originally presented over the Columbia Broadcasting System.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR

The Bronx High School of Science, New York City

1. Blank, Laura, "Seeking Formulas for Tables of Related Values." *School Science and Mathematics*, 39: 867-869. December, 1939.

The author lists seven tables of related values that may be useful as suitable exercises in discovering formulas of the first degree. It is suggested, however, that a few tables be given which will yield quadratic formulas or perhaps no formula at all in order to teach the students a little mathematical humility as well as prevent the growth of the notion that all relationships are linear in character.

2. Casner, Sidney, and Nyberg, Joseph A., "What Do High School Graduates Know about Arithmetic?" *The Journal of Business Education*, Vol. 15, No. 1, pp. 17-18. September, 1939.

The authors describe a test in arithmetic which they gave at Hyde Park High School in Chicago, Illinois, in connection with work in vocational guidance to obtain some information about the arithmetical abilities of the graduating seniors. The items of the test are analyzed and a summary of the results is presented. The following are some of the conclusions and recommendations:

1. All departments of the high school should co-operate in giving training in solving arithmetic problems that may arise in life situations, although the mathematics department should be chiefly responsible.
2. A course in business or economic or social arithmetic open to junior or senior pupils should be recommended to all those pupils who either are not planning to continue with their formal education beyond the high school or who are planning to continue along commercial lines.
3. This course should offer instruction not only in what might be called strictly arithmetical subject-matter, but also in what is known professionally as intuitive geometry.
3. Nygaard, P. H., "Mathematics and Democracy," *School Science and Mathematics*, 39: 847-853. December, 1939.

An interesting analysis of the role that mathematical ideals, concepts, and operations play in a democratic form of government.

4. Read, C. B., and Hitt, J. K., "Undefined Expressions Involving Fractional Exponents." *School Science and Mathematics*, 39: 839. December, 1939.

Most textbooks in algebra state the definition of a symbol having a fractional exponent in the following way:

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Many, however, fail to point out the important restriction on this definition that if n is an even integer, a must be positive. The resulting ambiguities are pointed out.

5. Reeve, W. D. "Dr. Breslich as a Leader in Secondary School Mathematics." *Zeta News*, Vol. 25, No. 1, December, 1939, pp. 1-2, 5.

An appreciative but restrained and analytical evaluation of the contribution of Dr. Breslich to the teaching of secondary mathematics in America. His influence is treated under the following categories:

- a. "The effect of his work on his pupils in the University High School for the past twenty-five years.
- b. "The influence that he has exerted both by example and by precept upon the faculty of his department during that time.
- c. "The modifications in their practice of scores of teachers of mathematics and others who have had the opportunity and the privilege of visiting Professor Breslich's classroom over the long period of time he has served the University of Chicago.
- d. "The influence of his secondary textbooks upon the organization of materials and upon classroom practice throughout this country for more than a quarter of a century.
- e. "The effect upon college and university students' thinking and subsequent practice of Professor Breslich's books on administration and teaching of mathematics. . . .

- f. "The reform which has been made by Professor Breslich through his personal contacts with organizations of teachers of mathematics by giving them words of counsel in public addresses, in work of important committees, and in magazine articles."

In the same issue of the magazine, there are two more articles about Dr. Breslich: One by William C. Reavis on "Dr. Breslich as a Member of the Department of Education," and a second by George E. Hawkins, entitled "Dr. Breslich as a Teacher in University High School."

A partial bibliography of Dr. Breslich's work, his vita, and photograph are also contained in this issue.

6. Shreve, D. R., and Keller, M. W., "A Note on the Sketching of the General Parabola." *School Science and Mathematics*, 39: 812-813. December, 1939.

"The purpose of this note is to recall to teachers of analytic geometry a simple and accurate method of constructing the general parabola whose equation contains the xy term. A simple application of composition of ordinates replaces the usually unsatisfactory rotation of axes without introducing any unfamiliar concepts."

7. Tweedie, M. C. K., "A Graphical Method of Solving Tartaglian Measuring Puzzles." *The Mathematical Gazette*, 23: 278-282. July, 1939.

The Tartaglian measuring puzzles are of the following familiar type: A man has three vessels, whose capacities are 3, 5, and 8 pints respectively. The largest is full of water. He desires to divide this water into two equal parts by using these vessels only. What are the simplest ways of doing this?

The usual method of solving such problems is that of trial and error, a method not only wasteful of time and energy but also contrary to the spirit of the mathematician. The author presents an elegant, graphical method of solving such problems and also elaborates on the theory underlying it.

8. Ulmer, Gilbert, "Teaching Geometry to Cultivate Reflective Thinking: An Experimental Study with 1239 High School Pupils." *The Journal of Experimental Education*, 8: 18-25. September, 1939.

The purpose of the study was "to evaluate the results obtained by a number of high school geometry teachers in different localities who used a method of teaching geometry in which emphasis was placed upon the cultivation of reflective thinking."

The procedure of the experiment is carefully described and the results presented in the familiar tabular forms.

Emphasis was placed upon the following factors of reflective thinking:

- a. If-then or postulational thinking
- b. The importance of defining key words and phrases
- c. Reasoning by generalization
- d. Reasoning by analogy
- e. Detecting implicit assumptions
- f. Inverses and converses
- g. Indirect proof
- h. Name calling.

The test items concerned themselves chiefly with items a , b , g , and h , and to a lesser degree with d .

The author concludes that the results of his study "indicate that it is possible for high school geometry teachers, under normal classroom conditions to teach in such a way as to cultivate reflective thinking, that this can be done without sacrificing an understanding of geometric relationships, and that pupils at all I.Q. levels are capable of profiting from such instruction. The results also indicate that even what is commonly regarded as superior geometry teaching has little effect upon pupils' behavior in the directions of reflective thinking unless definite provisions are made to study methods of thinking as an important end of itself."

9. Yates, Robert C., "A Rose Linkage, Trisection, and the Regular Heptagon." *School Science and Mathematics*, 39: 870-892. December, 1939.

In an article that appeared in *The American Mathematical Monthly* in 1932, Vol. 39, pp. 230-231, R. K. Morley pointed out the trisecting characteristics of the Four-Leaf Rose. In this article, the present author describes the construction of a linkage which will trace this curve and which, of course, may act directly as a trisector itself. The application of linkage to the construction of a regular heptagon is also indicated.

The mathematical theory underlying the construction and three diagrams are also included.

NEWS NOTES

Important news for all Council members comes from Oregon. In December 1938, our Oregon Representative, Edgar E. De Cou, gave an address on "The Work and Publications of the National Council of Teachers of Mathematics" before the Science and Mathematics Section of the Oregon State Teachers Association. Following the address a committee of three was appointed to organize an affiliation with the National Council. Organization was effected February 18, 1939, the petition for affiliation was sent to Secretary-Treasurer Edwin W. Schreiber, and the charter of affiliation to the Portland Council of Teachers of Mathematics was presented by Mr. De Cou at a large meeting held in Reed College, Portland.

The Portland Council, which includes the mathematics teachers of Portland and the vicinity, consists of one-third of the mathematics teachers in Oregon. The elected officers are: President, Miss Lesta Hoel, Grant High School, Portland; Vice-President, Olin L. Wills, Lincoln High School, Portland; Recording Secretary, Mrs. Genevieve Simpson, Grant High School, Portland, who is also local editor for *THE MATHEMATICS TEACHER*; Secretary-Treasurer, Wayne L. Bauer, Oregon Senior High School, Oregon City.

Good luck Portland Council! May your excellent example be followed by mathematics groups in other states!

Mathematics section meetings were held in five of the six districts as a part of the annual sessions of the Nebraska State Teachers Association on October 25-27. The following programs were presented.

DISTRICT NO. 1 AT LINCOLN

Division Meeting—Mathematics, Mental Hygiene, Physical Education and Science.

Chairman, K. O. Turner, Waverly; Secretary, Josephine Wible, Lincoln.

Program

Symposium—"Contributions of Mathematics, Mental Hygiene, Physical Education and Science to Economic Efficiency and Civic Responsibility."

Introductory Remarks—Dr. F. E. Henzlik, Dean, Teachers College, University of Nebraska.

Panel Discussion: Mathematics—Dr. A. R. Congdon, University of Nebraska; Josephine Wible, Lincoln High School; Mental Hygiene—

Prof. Roy Deal, Nebraska Wesleyan University; Clara M. Slade, psychologist, Lincoln City Schools; Physical Education—Robert V. Chase, College View High School, Lincoln; Dwight Thomas, Nebraska Wesleyan University; Science—Wayne Nicholls, Fairmont; and K. O. Turner, Waverly.

Section Meeting: President, Josephine Wible, Lincoln; Vice-President, H. B. Kliever, Henderson; Secretary, Jessamine Fugate, Beatrice.

"The Effect of Improvement of Reading upon Achievement in Plane Geometry," Ruth E. Callender, York.

"A Lesson on the Parabola," Dr. C. C. Camp, University of Nebraska.

DISTRICT NO. 2 AT OMAHA

Division Meeting: Administration, Visual Education, Science, Mathematics.

"The Rise of Science in the Modern School," Dr. Francis D. Curtis, University of Michigan.

DISTRICT NO. 3 AT NORFOLK

Section Meeting: Chairman, Leo R. Taylor, Norfolk. "High School Mathematics," William Lenser, Neligh; "Methods in Dealing with the Above Average and the Below Average Students in Mathematics," Dr. James M. Earl, University of Omaha.

Visual Demonstration.

DISTRICT NO. 4 AT HASTINGS

Section Meeting: President, Homer Rector, Ogallala; Vice-President, Harold Mueller, Nichols; Secretary, Milton Beckmann, Kearney. "Building a Workable Curriculum for the Teaching of Mathematics in the Nebraska Junior and Senior High Schools," R. M. Mc Dill, Hastings College.

Joint Science and Mathematics Section Meeting: "Practical Suggestions for Teaching Dull Normal Pupils Science and Mathematics," Dr. F. D. Curtis, University of Michigan.

DISTRICT NO. 5 AT HOLDREGE

No mathematics section meeting was held in this district.

DISTRICT NO. 6 AT SIDNEY

President, Gwendolyn Jorgenson, Sidney; Vice-President, D. E. McDonald, Belmont. "Report of Mathematics Meeting," Lena Meyer, Kimball; "Discussion of the Teaching

of Geometry"—Talks by selected teachers on the various phases of the techniques of teaching geometry.

A review of the complete programs held in the various districts shows that in some of the non-mathematics section meetings some time was given to topics of mathematics interest. Among these was "Graphs" by Mrs. Elsie Henthorn of Lincoln at Lincoln in the primary-elementary section in connection with the general topic of "Directing Learning Through the Use of Visual Materials," a series of demonstration lessons conducted by elementary school teachers and pupils.

ARTHUR L. HILL

The Mathematics Section of the High School Conference (the Illinois Association of the National Council of Teachers of Mathematics) met on November 3 in the Mathematics Building on the University of Illinois campus, Urbana. The theme of the program was "Arithmetic—an Art and a Science." The meeting was called to order by the chairman, Miss Jessie D. Brakensiek.

The first speaker of the morning session was R. O. Gibbons, who spoke on "Arithmetic in the Changing Elementary Schools of Today," considering the problem from the standpoint of content, grade placement, and methods.

Edgar W. Schreiber made an announcement on behalf of the National Council of Teachers of Mathematics and the Fourteenth Yearbook.

W. B. Storm of Northern Illinois State Teachers College next spoke on "The Modern Mathematical Background and Training of the Elementary School Teacher," quoting studies by E. H. Taylor of Charleston and others; and pointing out as remedies for poor preparation for elementary school mathematics teachers: (1) more college mathematics courses for people preparing to teach, (2) more arithmetic courses in high school, (3) better pre-school training.

T. A. Nelson of Decatur announced the annual convention of the Central Association of Science and Mathematics Teachers at the Morrison Hotel, Chicago, Illinois, November 24, 25.

The chairman announced the appointment of the nominating committee as follows: Miss Martha Hildebrandt, chairman, L. E. Mensenkamp, Dr. Henrietta Terry.

Edgar W. Schreiber of Western Illinois State Teachers College spoke on "The Dilemma of the High School Teacher," giving some very helpful suggestions as to new ways of teaching old material.

Dr. Miles Hartley of the University High

School led the discussion. Among those who contributed were:

E. H. Taylor who said meanings should be taught early.

Miss Hildebrandt who said we should teach more arithmetic with algebra and make it spontaneous.

H. G. Ayre who gave more evidence of the need for better mathematics teaching.

Dr. Miles Hartley was in charge of the noon luncheon. He introduced Dr. Crathorne who spoke on behalf of the University faculty.

The afternoon session was called to order at 2:00 P.M. The minutes were read and approved. The nominating committee named the following officers for the next year: Chairman, H. G. Ayre, Macomb; Vice-Chairman, Anice Seybold, Monticello; Secretary, Frances Innes, Dundee. The report of the committee was accepted.

Miss Ruth Lane of the University High School of the State University of Iowa spoke on "Leadership by High School Mathematics Teachers," showing how the individual teacher should assume leadership in the classroom, community, state, and nation.

Dr. D. W. Kerst of the Physics Department of the University of Illinois explained the meaning of the new M. K. S. system to be adopted by engineers on January 1, 1940, and spoke on "Teaching Mathematics Today for the World of Tomorrow." He stated that the frontiers of science, now, will eventually be high school textbook material. This situation will call for rearranging of courses to prevent cluttering. Some of the most noticeable mathematical needs of physics students are: the ability to tackle a problem, the habit of using letters to save numerical calculation, understanding of variation and knowledge of radians.

The rest of the meeting was given over to a play "Mathematics for the Millions" by Milton Kaletsky, presented by students of the University High School, Urbana, Illinois, under the direction of Henrietta Terry.

ANICE SEYBOLD

The Iowa Association of Mathematics Teachers met at Des Moines, Friday, November 3.

PROGRAM

Luncheon in the church basement at 12:15 P.M.

Luncheon address: "Fun in Geometry," Miss Vivian Strand, Red Oak Junior College.

Afternoon meeting in second floor auditorium at 2:00 P.M.

"Units in Numerical Mathematics for the Twelfth Grade," Joe Kirkman, Stuart.

"Mathematical Needs of Industrial Arts,"
Edwin C. Dethlefs, Le Mars.

Business meeting.

Report of the Curriculum Committee, Dr.
Van Engen, Iowa State Teachers College.

"A Fuller Account of the Causes of Failure,"
Dr. F. B. Knight, Purdue University, Lafayette,
Indiana.

RUTH LANE, *Secretary*

The Study Club of Elizabeth, New Jersey, mathematics teachers was organized early in 1938 for the purpose of studying the Tentative Report of the Joint Commission. It was made up of twenty-two of the forty secondary-school mathematics teachers in Elizabeth. Dr. Amanda Loughren, head of the mathematics departments in the Elizabeth public schools is an active member and sponsor.

The 1939 season began with a "Get-Acquainted Tea" given by the Study Club for all the secondary-school teachers of mathematics. The group has grown to nearly fifty members and includes several teachers from the elementary schools. Our topic for study is "Problem Solving in Life," and fifteen meetings have been planned. Dr. Drushell is the leader at the present time.

EVELYN K. TIEGER, *Chairman*

Dr. Aaron Bakst addressed Section 19 (mathematics) of the New York Society for the Experimental Study of Education on "The Application of Graphical Methods in Approximate Computation to Geometry" at the Men's Faculty Club of Columbia University, on the evening of December 2.

Professor John R. Clark, of Teachers College, Columbia University, is the new chairman of this group.

The Texas State Teachers Association held its 61st Annual Convention at San Antonio, November 30 through December 2.

Dr. Eric T. Bell, head of the department of mathematics, California Institute of Technology, and author of popular books on mathematics, addressed the general session of the Convention on "What Mathematics Can Do for Human Beings." He also addressed the Arith-

metic Section and the Mathematics Section on "Mathematics in the Education of Intelligent Citizens."

Other speakers for the Arithmetic Section were Mrs. Mary Koetter of San Antonio, who conducted a "High Fifth Grade Arithmetic Class Demonstration," and Miss Mary Ruth Cook, speaking on "What Is New in the Teaching of Arithmetic?"

The Men's Mathematics Club of Chicago and the Metropolitan Area met on November 17 for dinner and to hear M. S. Lindberg on "Some Observations on General Mathematics in the High School."

Professor E. R. Breslich, recently retired from the University of Chicago after thirty-five years of service, gave an address entitled "Mathematics."

The 366th meeting of the American Mathematical Society will be held at Columbia University, New York City, on Saturday, February 24.

By invitation of the Program Committee, Professor B. O. Koopman of Columbia University will deliver an address entitled "The Bases of Probability."

Abstracts of papers to be presented at this meeting should be in the hands of the undersigned not later than January 27.

T. R. HOLLCROFT, *Associate Secretary*

The 53rd Annual Convention of the Middle States Association of Colleges and Secondary Schools and Affiliated Associations was held in Atlantic City, New Jersey, on November 24 and 25. General meetings on the first day of the convention were followed by meetings of affiliated associations on the second day. The Association of Teachers of Mathematics of the Middle States presented the following program at its meeting on the twenty-fifth.

"Pitfalls in the Teaching of Mathematics," George E. Stetson, West Nottingham Academy; "A Brief Visit to the Schools of Rio de Janeiro," Miss Ruth Wyatt, Woodrow Wilson Junior High School, Philadelphia; and "The Basic Ideas of Arithmetic and Algebra," Dr. Francis D. Murnaghan, Johns Hopkins University.